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STEADY VIBRATIONS OF WING OF CIRCULAR PLAN FORM*

and

THEORY OF WING OF CIRCULAR PLAN FORM**

By N. E. Kochin

Translation

*"Ob ustanovivshikhsya kolebaniyakh kryla krugovoi formy v plane." Prikladnaya Matematika i Mekhanika, Vol. VI, 1942.

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The nonvortical motion of an ideal incompressible fluid has been solved (reference 1) for the case of uniform rectilinear motion of a wing of circular plan form. The method developed in reference 1 may also be generalized to the case of the nonsteady motion of such wing. The problem of the steady vibrations of a circular wing is solved herein. The results will be frequently referred to herein. The problem of the steady vibrations of a circular wing was solved by another method by Th. Schade (reference 2).

1. Fundamental equations

The wing, the motion of which is under consideration, is assumed, as in reference 1, to be thin and slightly curved; its projection on the xy-plane has the shape of a circle ABCD of radius a with center at the origin of coordinates. The principal motion of the wing is assumed to be a rectilinear translational motion with constant velocity c parallel to the x-axis. The coordinate axes are assumed as displaced with the same velocity. On the principal motion of the wing is superposed its additional harmonic vibration of frequency ω, where the possibility of deformation of the wing is not excluded. The equation of the surface of the wing may then be represented in the form:

$$z(x,y,t) = \zeta_0(x,y) + \zeta_1(x,y) \cos \omega t + \zeta_2(x,y) \sin \omega t \qquad (1.1)$$

where the ratios ζ_k/a as well as the derivatives $\partial\zeta_k/\partial x$ and $\partial\zeta_k/\partial y$, where $k = 0,1,2$, are assumed small magnitudes.

The fluid is assumed incompressible and the motion is assumed nonvortical and occurring in the absence of external forces. The velocity

*"Ob ustanovivshikhsya kolebaniyakh kryla krugovoi formy v plane." Prikladnaya Matematika i Mekhanika, Vol. VI, 1942, pp. 287-316.

potential will be denoted by $\varphi(x,y,z,t)$ and steady vibrations of the fluid will be assumed; that is, the velocity potential is represented in the form:

$$\varphi(x,y,z,t) = \varphi_0(x,y,z) + \varphi_1(x,y,z)\cos\omega t + \varphi_2(x,y,z)\sin\omega t$$

It is evident that the functions φ_0, φ_1, and φ_2 satisfy the equations of Laplace

$$\frac{\partial^2\varphi_k}{\partial x^2} + \frac{\partial^2\varphi_k}{\partial y^2} + \frac{\partial^2\varphi_k}{\partial z^2} = 0 \qquad (k = 0,1,2)$$

The velocity of the particles of the fluid near the leading edge of the wing DAB is assumed to approach infinity as $\delta^{-1/2}$, where δ is the distance of the particle from the leading edge, but the velocity of the fluid particles near the trailing edge of the wing BCD is assumed as finite. From this edge a surface of discontinuity passes off on which the function φ undergoes a discontinuity. As in reference 1, the problem will be linearized. Since the values of the functions φ_k and their derivatives are assumed to be small quantities of the first order, their squares and products are rejected. The functions $\varphi_k(x,y,z)$ are further assumed to have discontinuities on the infinite half-strip Ξ situated in the xy-plane in the direction of the negative x-axis from the rear semicircumference BCD of the circle S to infinity. The boundary conditions on the surface of the wing are replaced by the conditions on the circle S located in the xy-plane. Everywhere outside the half-strip Ξ and the circle S the functions $\varphi_k(x,y,z)$ are thus regular functions.

The boundary conditions which these functions satisfy are now set up. On the surface of discontinuity Ξ, the kinematic condition expressing the continuity of the normal component of the velocity must first of all be satisfied:

$$\left(\frac{\partial\varphi}{\partial z}\right)_{z=+0} = \left(\frac{\partial\varphi}{\partial z}\right)_{z=-0}$$

from which is obtained the conditions

$$\left(\frac{\partial\varphi_k}{\partial z}\right)_{z=+0} = \left(\frac{\partial\varphi_k}{\partial z}\right)_{z=-0} \qquad \text{on } \Xi \tag{1.2}$$

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The dynamical conditions expressing the continuity of the pressure in passing through the surface of discontinuity Ξ are now stated.

If a stationary system of coordinates $x_1 y_1 z_1$ is employed, connected with the coordinates xyz of the moving system of coordinates by the relations

$$x = x_1 - ct_1 \qquad y = y_1 \qquad z = z_1 \qquad t = t_1$$

then the pressure may be determined by the following formula:

$$p = -\rho \frac{\partial \varphi}{\partial t_1} - \frac{\rho}{2}\left[\left(\frac{\partial \varphi}{\partial x_1}\right)^2 + \left(\frac{\partial \varphi}{\partial y_1}\right)^2 + \left(\frac{\partial \varphi}{\partial z_1}\right)^2\right] + F(t_1) \tag{1.3}$$

Since

$$\frac{\partial \varphi}{\partial t_1} = \frac{\partial \varphi}{\partial t} - c\frac{\partial \varphi}{\partial x} \qquad \frac{\partial \varphi}{\partial x_1} = \frac{\partial \varphi}{\partial x} \qquad \frac{\partial \varphi}{\partial y_1} = \frac{\partial \varphi}{\partial y} \qquad \frac{\partial \varphi}{\partial z_1} = \frac{\partial \varphi}{\partial z} \tag{1.4}$$

the following equation will apply in the movable xyz system:

$$p = -\rho \frac{\partial \varphi}{\partial t} + \rho c \frac{\partial \varphi}{\partial x} - \frac{\rho}{2}\left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2\right] + F(t) \tag{1.5}$$

When small quantities of the second order are rejected and the magnitude $F(t)$ is not dependent on the coordinates,

$$p = -\rho \frac{\partial \varphi}{\partial t} + \rho c \frac{\partial \varphi}{\partial x}$$

or, on account of equation (1.2),

$$p(x,y,z,t) = \rho c \frac{\partial \varphi_0}{\partial x} + \left(\rho c \frac{\partial \varphi_1}{\partial x} - \rho\omega\varphi_2\right) \cos \omega t + \left(\rho c \frac{\partial \varphi_2}{\partial x} + \rho\omega\varphi_1\right) \sin \omega t \tag{1.6}$$

For briefness, the following notation is introduced:

$$\omega/c = k \tag{1.7}$$

The condition of continuity of the pressure on Ξ then leads to the three equations:

$$\left(\frac{\partial\varphi_0}{\partial x}\right)_{z=+0} \quad \left(\frac{\partial\varphi_0}{\partial x}\right)_{z=-0}$$

$$\left(\frac{\partial\varphi_1}{\partial x} - k\varphi_2\right)_{z=+0} = \left(\frac{\partial\varphi_1}{\partial x} - k\varphi_2\right)_{z=-0} \quad \text{on } S \qquad (1.8)$$

$$\left(\frac{\partial\varphi_2}{\partial x} + k\varphi_1\right)_{z=+0} = \left(\frac{\partial\varphi_2}{\partial x} + k\varphi_1\right)_{z=-0}$$

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The condition on the circle S is now written. Equation (1.1) in the stationary system of coordinates has the form:

$$z_1 = \zeta_0(x_1-ct_1,y_1) + \zeta_1(x_1-ct_1,y_1) \cos \omega t_1 + \zeta_2(x_1-ct_1,y_1) \sin \omega t_1$$

Hence, for the normal component of the velocity of the fluid particles adjacent to the surface of the wing,

$$\frac{dz_1}{dt_1} = - c \frac{\partial\zeta_0}{\partial x} - c \frac{\partial\zeta_1}{\partial x} \cos \omega t - \omega\zeta_1 \sin \omega t - c \frac{\partial\zeta_2}{\partial x} \sin \omega t + \omega\zeta_2 \cos \omega t$$

The notations

$$- c \frac{\partial\zeta_0}{\partial x} = Z_0(x,y) \qquad - c\left(\frac{\partial\zeta_1}{\partial x} - k\zeta_2\right) = Z_1(x,y) \qquad - c\left(\frac{\partial\zeta_2}{\partial x} + k\zeta_1\right) = Z_2(x,y)$$

yield the boundary condition

$$\left(\frac{\partial\varphi}{\partial z}\right)_{z=0} = Z_0(x,y) + Z_1(x,y) \cos \omega t + Z_2(x,y) \sin \omega t$$

which must be satisfied on both the upper and the lower sides of the circle S and which breaks own into the three conditions:

$$\left(\frac{\partial\varphi_k}{\partial z}\right)_{z=0} = Z_k(x,y) \quad \text{on } S \quad (k = 0,1,2) \qquad (1.9)$$

The presence of conditions (1.2) and (1.9) permits consideration of the functions $\varphi_k(x,y,z)$ as odd functions of Z:

$$\varphi_k(x,y,-z) = - \varphi_k(x,y,z) \qquad (1.10)$$

If it is assumed, in particular, that $z = 0$,

$$\varphi_k(x,y,0) = 0 \tag{1.11}$$

in the entire xy-plane with the exception of the circle S and the half strip Σ on which φ_k undergoes a discontinuity.

The conditions (1.8), because of equation (1.10), assume the form:

$$\frac{\partial\varphi_0}{\partial x} = 0 \qquad \frac{\partial\varphi_1}{\partial x} - k\varphi_2 = 0 \qquad \frac{\partial\varphi_2}{\partial x} + k\varphi_1 = 0 \qquad \text{on } \Sigma \tag{1.12}$$

Finally, the absence of a disturbance of the fluid far ahead of the wing leads to the evident conditions at infinity:

$$\lim_{x\to+\infty} \frac{\partial\varphi_k}{\partial x} = \lim_{x\to+\infty} \frac{\partial\varphi_k}{\partial y} = \lim_{x\to+\infty} \frac{\partial\varphi_k}{\partial z} = 0 \tag{1.13}$$

The problem of determining the function $\varphi_0(x,y,z)$ satisfying all obtained conditions for this function was considered in reference 1.

The following equality is set up:

$$\Phi(x,y,z) = \varphi_1(x,y,z) + i\varphi_2(x,y,z) \tag{1.14}$$

so that

$$\varphi(x,y,z,t) = \varphi_0(x,y,z) + \mathrm{Re}\left\{\Phi(x,y,z)e^{-i\omega t}\right\} \tag{1.15}$$

Also,

$$\zeta_1(x,y) + i\zeta_2(x,y) = \zeta(x,y)$$

$$Z_1(x,y) + iZ_2(x,y) = Z(x,y) = -c\left(\frac{\partial\zeta}{\partial x} + ik\zeta\right) \tag{1.16}$$

The shape of the wing will be determined by the equation

$$z(x,y,t) = \zeta_0(x,y) + \mathrm{Re}\left\{\zeta(x,y)e^{-i\omega t}\right\} \tag{1.17}$$

The functions $\Phi(x,y,z)$ will then be a harmonic function, regular in the entire half-space $z > 0$ and satisfying the conditions:

$$\left(\frac{\partial\Phi}{\partial z}\right)_{z=0} = Z(x,y) \quad \text{on } S \tag{1.18}$$

$$\frac{\partial\Phi}{\partial x} + ik\Phi = 0 \quad \text{on } \Sigma \tag{1.19}$$

following from equations (1.9) and (1.12). In the entire remaining part of the plane xy the following condition must be satisfied:

$$\Phi(x,y,0) = 0 \tag{1.20}$$

Moreover, the following conditions must be satisfied at infinity:

$$\lim_{x\to+\infty} \frac{\partial\Phi}{\partial x} = \lim_{x\to+\infty} \frac{\partial\Phi}{\partial y} = \lim_{x\to+\infty} \frac{\partial\Phi}{\partial z} = 0 \tag{1.21}$$

which are the boundary conditions of the first derivatives of the function $\Phi(x,y,z)$ near the rear semicircumference BCD of the circle S and the condition that near the forward semicircumference DAB these derivatives may become infinite to the order of $\delta^{-1/2}$.

2. Fundamental formulas

In reference 1 an expression was constructed, which depended on an arbitrary function $f_0(x,y)$, which determined a harmonic function $\varphi_0(x,y,z)$ satisfying all the conditions imposed in the preceding section

$$\varphi_0(x,y,z) = \frac{1}{2\pi} \iint_S f_0(\xi,\eta) \Bigg\{K(x,y,z,\xi,\eta) + \frac{1}{\pi^2\sqrt{2}} \int_{+\infty}^{x} \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{G(x,y,z,\gamma)\sqrt{a^2 - \xi^2 - \eta^2} \cos\gamma \, d\gamma \, dx}{(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)}\Bigg\} d\xi\, d\eta \tag{2.1}$$

The functions $K(x,y,z,\xi,\eta)$ and $G(x,y,z,\gamma)$ for $z > 0$ are given by

$$K(x,y,z,\xi,\eta) = \frac{2}{\pi r} \arctan \frac{\sqrt{a^2 - \xi^2 - \eta^2}\sqrt{a^2 - x^2 - y^2 - z^2 + R}}{\sqrt{2}\, ar}$$

$$G(x,y,z,\gamma) = \frac{\sqrt{a^2 - x^2 - y^2 - z^2 + R}}{x^2 + y^2 + z^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma} \tag{2.2}$$

which are harmonic functions of x, y, z where

$$
\begin{aligned}
r &= \sqrt{(x - \xi)^2 + (y - \eta)^2 + z^2} \\
R &= \sqrt{(a^2 - x^2 - y^2 - z^2)^2 + 4a^2z^2}
\end{aligned}
\tag{2.3}
$$

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In order to satisfy boundary condition (1.9)

$$
\frac{\partial \varphi_0}{\partial z} = Z_C(x,y) \qquad \text{on} \quad S \tag{2.4}
$$

it is necessary to take

$$
f_0(x,y) = - Z_C(x,y) + g_0(y) \tag{2.5}
$$

where $g_0(y)$ is determined from a Fredholm integral equation of the second kind.

The solution of the more general problem of steady vibrations may be presented in a similar form.

Thus, $f_1(x,y)$ and $f_2(x,y)$ denote two arbitrary real functions, continuous, together with their partial derivatives of the first and second order, in the entire circle S;

$$
f_1(x,y) + if_2(x,y) = f(x,y) \tag{2.6}
$$

It will now be shown that the function

$$
\Phi(x,y,z) = \frac{1}{2\pi} \iint_S f(\xi,\eta) \left\{ K(x,y,z,\xi,\eta) + \frac{1}{\pi^2\sqrt{2}} e^{-ikx} \int_{+\infty}^{x} \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{G(x,y,z,\gamma)\, e^{ikx} \sqrt{a^2 - \xi^2 - \eta^2}\ [illegible]\ d\gamma\, dx}{(\xi^2 + \eta^2 + a^2 - 2a\xi \cos \gamma - 2ay \sin \gamma)} \right\} d\xi\, d\eta \tag{2.7}
$$

satisfies all the conditions of the preceding section except condition (1.18).

The function $G(x,y,z,\gamma)$, as shown by equation (2.2), is harmonic; hence the function

$$L(x,y,z) = e^{-ikx}\int_{+\infty}^{x} e^{ikx}\, G(x,y,z)\, dx$$

will be a harmonic function. In fact,

$$\Delta L = \frac{\partial^2 L}{\partial x^2} + \frac{\partial^2 L}{\partial y^2} + \frac{\partial^2 L}{\partial z^2} = \frac{\partial G}{\partial x} - ikG + e^{-ikx}\int_{+\infty}^{x} e^{ikx}\left[\frac{\partial^2 G}{\partial y^2} + \frac{\partial^2 G}{\partial z^2} - k^2 G\right] dx$$

$$= \frac{\partial G}{\partial x} - ikG - e^{-ikx}\int_{+\infty}^{x} e^{ikx}\left(\frac{\partial^2 G}{\partial x^2} + k^2 G\right) dx$$

When this expression is integrated by parts, it is easily shown that $\Delta L = 0$, since both G and $\partial G/\partial x$ approach zero for $x \to \infty$.

It then follows that the function $\Phi(x,y,z)$ likewise satisfies the Laplace equation

$$\Delta\Phi = 0 \tag{2.8}$$

where from the form of equation (2.7) it is seen that $\Phi(x,y,z)$ is regular everywhere outside the circle S and half strip Ξ. In exactly the same way it is shown that the conditions at infinity (1.21) and condition (1.20) are satisfied.

Furthermore,

$$\frac{\partial\Phi}{\partial x} + ik\Phi = \frac{1}{2\pi}\iint_S f(\xi,\eta)\left\{\frac{\partial K}{\partial x} + ikK + \right.$$

$$\left.\frac{1}{\pi^2\sqrt{2}}\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{G(x,y,z,\gamma)\sqrt{a^2 - \xi^2 - \eta^2}\cos\gamma\, d\gamma}{(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)}\right\} d\xi\, d\eta \tag{2.9}$$

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It is clear that if $x^2 + y^2 > a^2$ then

$$\frac{\partial\Phi}{\partial x} + ik\Phi = 0 \qquad \text{for} \quad z = 0 \tag{2.10}$$

so that condition (1.19) is likewise satisfied.

It thus remains to check the finiteness of the derivatives of the function $\Phi(x,y,z)$ at the points of the semicircumference BCD of the circle S and to establish the behavior of these derivatives near the forward semicircumference DAB. But near the forward semicircumference, the inside integral in formula (2.7) evidently remains bounded, as do its partial derivatives; since the first derivatives of the integral

$$\iint_S f(\xi,\eta)\, K(x,y,z,\xi,\eta)\, d\xi\, d\eta$$

as established in reference 1, and as will again be proven, have near the contour of the circle S the order $\delta^{-1/2}$ (where δ is the distance of a point to the contour ABCD of the circle S), it is clear that the first derivatives of the function $\Phi(x,y,z)$ also have the order $\delta^{-1/2}$ near the forward semicircumference DAB of the circle S.

For determining the behavior of the function $\Phi(x,y,z)$ near the rear semicircumference BCD, the right side of equation (2.9) is transformed. Denoting it by $M(x,y,z)$ and making use of formula (2.11) of reference 1 and the formula of integration by parts (2.14) of reference 1,

$$M(x,y,z) = \frac{1}{2\pi} \iint_S f(\xi,\eta) \left\{ ikK - \right.$$

$$\left. \frac{1}{\pi^2\sqrt{2}} \int_{-\frac{1}{2}\pi}^{+\frac{3}{2}\pi} \frac{G(x,y,z,\gamma)\sqrt{a^2-\xi^2-\eta^2}\cos\gamma\, d\gamma}{(\xi^2+\eta^2+a^2-2a\xi\cos\gamma-2a\eta\sin\gamma)} \right\} d\xi\, d\eta \quad \frac{1}{2\pi} \iint_S \frac{\partial f}{\partial \xi} K\, d\xi\, d\eta \tag{2.11}$$

It is evident that this function remains finite near the rear semicircumference BCD. But when the following equation is integrated,

$$\frac{\partial\Phi}{\partial x} + ik\Phi = M(x,y,z) \tag{2.12}$$

there is obtained

$$\Phi(x,y,z) = e^{-ikx}\int_0^x e^{ikx} M(x,y,z)\,dx + \Phi(0,y,z)\,e^{-ikx} \tag{2.13}$$

whence it is clear that both the function Φ and its derivative with respect to x remain finite near the rear semicircumference BCD. The derivatives of M with respect to y and z will be of the order $\delta^{-1/2}$ near BCD, as follows from a consideration analogous to that which was adduced previously for determining the behavior of the function $\Phi(x,y,z)$ near the forward edge of the wing DAB. Since

$$\frac{\partial\Phi}{\partial y} = e^{-ikx}\int_0^x e^{ikx}\frac{\partial M}{\partial y}\,dx + \frac{\partial\Phi(0,y,z)}{\partial y}\,e^{-ikx}$$

it is clear that the derivative $\partial\Phi/\partial y$, and similarly $\partial\Phi/\partial z$, remain finite near the rear edge of the wing BCD.

The function (2.7) thus satisfies all the imposed conditons. The only condition not utilized was condition (1.18)

$$\left(\frac{\partial\Phi}{\partial z}\right)_{z=0} = Z(x,y) \quad \text{on } S \tag{2.14}$$

When the following formulas are employed:

$$\lim_{z \to +0} \iint_S \frac{\partial K}{\partial z} f(\xi,\eta)\, d\xi\, d\eta = -2\pi f(x,y)$$

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$$\lim_{z \to +0} \frac{\partial}{\partial z}\sqrt{a^2 - x^2 - y^2 - z^2 + R} = \begin{cases} 0 & \text{for } x^2 + y^2 < a^2 \\ \dfrac{a\sqrt{2}}{\sqrt{x^2 + y^2 - a^2}} & \text{for } x^2 + y^2 > a^2 \end{cases}$$

it is found without difficulty that on S

$$\left(\frac{\partial \Phi}{\partial z}\right)_{z=0} = -f(x,y) + g(y)\, e^{-ikx} \tag{2.15}$$

where

$$g(y) = \frac{a}{2\pi^3} \iint_S \int_{+\infty}^{\sqrt{a^2-y^2}} \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{e^{ikx}\,(x^2 + y^2 - a^2)^{-1/2}\,(a^2 - \xi^2 - \eta^2)^{1/2} \cos\gamma\, f(\xi,\eta)\, d\gamma\, dx\, d\xi\, d\eta}{(x^2 + y^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)} \tag{2.16}$$

The following equation is thus obtained:

$$-f(x,y) + g(y)\, e^{-ikx} = Z(x,y) \tag{2.17}$$

whence

$$f(x,y) = -Z(x,y) + g(y)\, e^{-ikx} \tag{2.18}$$

Substitution of this value of the function $f(x,y)$ in equation (2.16) yields, for the determination of the function $g(y)$, an integral equation of Fredholm

$$g(y) = N(y) + \int_{-a}^{a} H(y,\eta)\, g(\eta)\, d\eta \tag{2.19}$$

where

$$N(y) = -\frac{a}{2\pi^3} \iint_S \int_{+\infty}^{\sqrt{a^2-y^2}} \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{e^{ikx}\sqrt{a^2 - \xi^2 - \eta^2}\; G(x,y,z,\gamma) \cos\gamma\, Z(\xi,\eta)\, d\gamma\, dx\, d\xi\, d\eta}{\sqrt{x^2 + y^2 - a^2}\,(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)} \tag{2.20}$$

with $G(x,y,z,\gamma)$ according to equations (2.2) and

$$H(y,\eta) =$$

$$\frac{a}{2\pi^3} \times \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} \int_{+\infty}^{\sqrt{a^2-y^2}} \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{e^{ik(x-\xi)}(x^2+y^2-a^2)^{-1/2}(a^2-\xi^2-\eta^2)^{1/2}\cos\gamma\, d\gamma\, dx\, d\xi}{(x^2+y^2+a^2-2ax\cos\gamma-2ay\sin\gamma)(\xi^2+\eta^2+a^2-2a\xi\cos\gamma-2a\eta\sin\gamma)} \qquad (2.21)$$

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3. Computation of forces

The pressure p may be determined from formula (1.6), which with the notation (1.14) may be written in the following form:

$$p = \rho c\left\{\frac{\partial\varphi_0}{\partial x} + \operatorname{Re}\left[\left(\frac{\partial\Phi}{\partial x} + ik\Phi\right) e^{-i\omega t}\right]\right\} \qquad (3.1)$$

For the computation of the forces acting on the wing, it is necessary to know the pressure on the circle S.

Because of equation (1.10), the pressures above and below the wing differ only in sign:

$$p_- = -p_+ \qquad (3.2)$$

For clarity, the signs of the functions on the wing will henceforth be assumed to be the limiting values in approaching the wing from above, that is, for $z \to +0$.

For the lift force P the following expression is obtained

$$P = \iint_S (p_- - p_+)\, dx\, dy = -2\iint_S p_+\, dx\, dy =$$

$$-2\rho c\iint_S \left\{\frac{\partial\varphi_0}{\partial x} + \operatorname{Re}\left[\left(\frac{\partial\Phi}{\partial x} + ik\Phi\right) e^{-i\omega t}\right]\right\} dx\, dy \qquad (3.3)$$

But by formulas (2.9) and (2.11), the following equation applies on the upper side of the circle S:

$$\frac{\partial \Phi(x,y,0)}{\partial x} + i\,k\,\Phi(x,y,0) = \frac{1}{2\pi}\iint_S \frac{\partial f}{\partial \xi} K(x,y,0,\xi,\eta)\,d\xi\,d\eta +$$

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$$\frac{1}{2\pi}\iint_S f(\xi,\eta)\Bigg[i\,k\,K(x,y,0,\xi,\eta) - \tag{3.4}$$

$$\frac{1}{\pi^2}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\sqrt{a^2 - x^2 - y^2}\,\sqrt{a^2 - \xi^2 - \eta^2}\,\cos\gamma\,d\gamma}{(x^2 + y^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)}\Bigg]d\xi\,d\eta$$

This expression is integrated over the entire area of the circle S. The order of integration is interchanged and the two integrals must be computed first of all by formula (4.13) of reference 1

$$\iint_S \frac{\sqrt{a^2 - x^2 - y^2}}{x^2 + y^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma}\,dx\,dy = 2\pi a \tag{3.5}$$

It will be proven further that

$$\iint_S K(x,y,0,\xi,\eta) = 4\sqrt{a^2 - \xi^2 - \eta^2} \tag{3.6}$$

For this proof, the following function is considered:

$$F(x,y,z) = \iint_S K(x,y,z,\xi,\eta)\,d\xi\,d\eta \tag{3.7}$$

Because of the definition of the function K, the function $F(x,y,z)$ is a harmonic function over the entire space outside the circle S. By formula (2.35) of reference 1, the following condition is satisfied on the surface of this circle:

$$\frac{\partial F}{\partial z} = -\,2\pi \quad \text{on} \quad S \tag{3.8}$$

and therefore the function (3.7) is the potential of the nonvortical motion of a fluid corresponding to the translational motion of a circular disk with velocity $+2\pi$ along the negative z-axis normal to the plane of the disk. This motion, however, belongs to those that have been studied in classical hydrodynamics, from which can be taken the corresponding expression of the function.

$$F(x,y,z) = \iint_S K(x,y,z,\xi,\eta)\, d\xi\, d\eta$$

$$= 2\sqrt{2}\sqrt{R + a^2 - x^2 - y^2 - z^2}\left\{1 - \sqrt{\frac{R + x^2 + y^2 + z^2 - a^2}{2a^2}}\ \text{arc ctn}\sqrt{\frac{R + x^2 + y^2 + z^2 - a^2}{2a^2}}\right\} \tag{3.9}$$

Passing to the limit $z \rightarrow +0$ yields the formula

$$\iint_S K(x,y,0,\xi,\eta)\, d\xi\, d\eta = 4\sqrt{a^2 - x^2 - y^2} \quad \text{on } S$$

which is equivalent to equation (3.6), since $K(x,y,0,\xi,\eta)$ is a symmetrical function with respect to the points $M(x,y)$ and $N(\xi,\eta)$.

The following formula is thus obtained:

$$\iint_S \left[\frac{\partial\Phi(x,y,0)}{\partial x} + i\, k\, \Phi(x,y,0)\right] dx\, dy = \frac{2}{\pi}\iint_S \sqrt{a^2 - \xi^2 - \eta^2}\left\{\frac{\partial f}{\partial \xi} + ikf\right\} d\xi\, dn -$$

$$\frac{a}{\pi^2}\iint_S \sqrt{a^2 - \xi^2 - \eta^2}\, f(\xi,\eta)\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos\gamma\, d\gamma}{\xi^2 + \eta^2 + a^2 - 2a\,\xi\cos\gamma - 2a\eta\sin\gamma}\, d\xi\, d\eta \tag{3.10}$$

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If this expression and similar expressions are substituted for the function φ_0, obtained from equation (3.10) for $k = 0$, the final expression of the lift force acting on the wing is obtained:

$$P = -\frac{4\rho c}{\pi}\iint_S \sqrt{a^2 - \xi^2 - \eta^2}\left\{\frac{\partial f_0}{\partial \xi} + \mathrm{Re}\left[e^{-i\omega t}\left(\frac{\partial f}{\partial \xi} + ikf\right)\right] - \right.$$

$$\left.\frac{a}{2\pi}\left[f_0 + \mathrm{Re}\left(e^{-i\omega t}\right)f\right]\int_{-\frac{1}{2}\pi}^{\frac{3}{2}\pi}\frac{\cos\gamma\, d\gamma}{\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma}\right\}d\xi\, d\eta \tag{3.11}$$

By integration by parts and with the aid of the following formula

$$\int_0^{2\pi}\frac{\cos\gamma\, d\gamma}{\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma} = \frac{2\pi\xi}{a(a^2 - \xi^2 - \eta^2)} \tag{3.12}$$

equation (3.11) may be rewritten in the form:

$$P = -\frac{4\rho c}{\pi}\iint_S \sqrt{a^2 - \xi^2 - \eta^2}\left\{\mathrm{Re}\left(ikfe^{-i\omega t}\right) + \right.$$

$$\left.\frac{a}{2\pi}\left[f_0 + \mathrm{Re}\left(fe^{-i\omega t}\right)\right]\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi}\frac{\cos\gamma\, d\gamma}{\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma}\right\}d\xi\, d\eta \tag{3.13}$$

In a similar manner, the formulas for the moments of the forces about the x- and y-axes are obtained.

For the moment of the pressure forces about the x-axis

$$M_x = \iint_S y(p_- - p_+)\, dx\, dy = -2\iint_S yp_+\, dx\, dy \tag{3.14}$$

there is obtained

$$M_x = -2\rho c \iint_S y\left\{\frac{\partial\varphi_0}{\partial x} + \mathrm{Re}\left[\left(\frac{\partial\Phi}{\partial x} + i\,k\,\Phi\right) e^{-i\omega t}\right]\right\} dx\,dy \tag{3.15}$$

The order of integration is interchanged by use of equation (3.4). It is here necessary to compute two integrals. By formula (4.44) of reference 1,

$$\iint_S \frac{y\sqrt{a^2 - x^2 - y^2}\,dx\,dy}{x^2 + y^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma} = \frac{4}{3}\pi a^2 \sin\gamma \tag{3.16}$$

It will now be shown that

$$\iint_S y\,K(x,y,0,\xi,\eta)\,dx\,dy = \frac{8}{3}\eta\sqrt{a^2 - \xi^2 - \eta^2} \tag{3.17}$$

For this derivation, the following function is considered:

$$F_1(x,y,z) = \iint_S \eta\,K(x,y,z,\xi,\eta)\,d\xi\,d\eta$$

By formula (2.35) of reference 1, the following equation applies on the circle S:

$$\frac{\partial F_1}{\partial z} = -2\pi y \tag{3.18}$$

and therefore $F_1(x,y,z)$ is the potential of the motion of a fluid corresponding to the rotation of a disk about the x-axis with angular velocity -2π, a case studied in classical hydrodynamics:

$$F_1(x,y,z) = \iint_S \eta\, K(x,y,z,\xi,\eta)\, d\xi\, d\eta$$

$$= 2\sqrt{2}\, y\sqrt{R + a^2 - x^2 - y^2 - z^2}\left\{1 - \frac{2a^2}{3(R + x^2 + y^2 + z^2 + a^2)} - \sqrt{\frac{R + x^2 + y^2 + z^2 - a^2}{2a^2}}\ \text{arc ctn}\sqrt{\frac{R + x^2 + y^2 + z^2 - a^2}{2a^2}}\right\} \qquad (3.19)$$

Passing to the limit $z \to +0$ yields the formula

$$\iint_S \eta\, K(x,y,0,\xi,\eta)\, d\xi\, d\eta = \frac{8}{3} y\sqrt{a^2 - x^2 - y^2} \qquad \text{on } S$$

equivalent to equation (3.17).

As a result, the following formula is obtained

$$\iint_S y\left[\frac{\partial\Phi}{\partial x} + i k \Phi\right] dx\, dy = \frac{4}{3\pi}\iint_S \eta\sqrt{a^2 - \xi^2 - \eta^2}\left\{\frac{\partial f}{\partial \xi} + ikf\right\} d\xi\, d\eta -$$

$$\frac{2a^2}{3\pi^2}\iint_S \sqrt{a^2 - \xi^2 - \eta^2}\, f(\xi,\eta)\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\sin\gamma\cos\gamma\, d\gamma}{\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma}\, d\xi\, d\eta \qquad (3.20)$$

Hence, for the moment of the pressure forces about the x-axis, the following expression is obtained:

$$M_x = -\frac{8\rho c}{3\pi}\iint_S \sqrt{a^2 - \xi^2 - \eta^2}\left\{\eta\left[\frac{\partial f_0}{\partial \xi} + \text{Re}\left\{e^{-i\omega t}\left(\frac{\partial f}{\partial \xi} + ikf\right)\right\}\right] - \right.$$

$$\left.\frac{a}{2\pi}\left[f_0 + \text{Re}(e^{-i\omega t} f)\right]\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\sin\gamma\cos\gamma\, d\gamma}{\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma}\right\} d\xi\, d\eta \qquad (3.21)$$

or, on account of the formula

$$\int_0^{2\pi} \frac{\sin\gamma \cos\gamma \, d\gamma}{\xi^2 + \eta^2 + a^2 - 2a\xi \cos\gamma - 2a\eta \sin\gamma} = \frac{2\pi\xi\eta}{a^2(a^2 - \xi^2 - \eta^2)} \tag{3.22}$$

the equivalent expression

$$M_x = -\frac{8\rho c}{3\pi} \iint_S \sqrt{a^2 - \xi^2 - \eta^2} \left\{ \eta \, \mathrm{Re}(ikfe^{-i\omega t}) + \frac{a^2}{2\pi}\left[f_0 + \mathrm{Re}(fe^{-i\omega t})\right] \times \right.$$

$$\left. \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{\sin\gamma \cos\gamma \, d\gamma}{\xi^2 + \eta^2 + a^2 - 2a\xi \cos\gamma - 2a\eta \sin\gamma} \right\} d\xi \, d\eta \tag{3.23}$$

In the same way for the moment of the pressure forces about the y-axis

$$M_y = -\iint_S x(p_- - p_+) \, dx \, dy = -2\iint_S xp_+ \, dx \, dy \tag{3.24}$$

there is obtained

$$M_y = 2\rho c \iint_S x\left\{\frac{\partial\varphi_0}{\partial x} + \mathrm{Re}\left[\left(\frac{\partial\Phi}{\partial x} + i\,k\,\Phi\right) e^{-i\omega t}\right]\right\} dx \, dy \tag{3.25}$$

It is here necessary to employ the formulas

$$\iint_S \frac{x\sqrt{a^2 - x^2 - y^2}}{x^2 + y^2 + a^2 - 2ax \cos\gamma - 2ay \sin\gamma} \, dx \, dy = \frac{4}{3}\pi a^2 \cos\gamma \tag{3.26}$$

$$\iint_S x\, K(x,y,0,\xi,\eta)\, dx\, dy = \frac{8}{3}\,\xi\sqrt{a^2 - \xi^2 - \eta^2} \tag{3.27}$$

As before, there is obtained

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$$M_y = \frac{8\rho c}{3\pi}\iint_S \sqrt{a^2 - \xi^2 - \eta^2}\left\{\xi\left[\frac{\partial f_0}{\partial \xi} + \mathrm{Re}\left\{e^{-i\omega t}\left(\frac{\partial f}{\partial \xi} + ikf\right)\right\}\right] - \right.$$

$$\left.\frac{a^2}{2\pi}\left[f_0 + \mathrm{Re}\left(e^{-i\omega t}\, f\right)\right]\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{\cos^2\gamma\, d\gamma}{\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma}\right\} d\xi\, d\eta \tag{3.28}$$

Integration by parts and use of the formula yields

$$\int_0^{2\pi}\frac{\cos^2\gamma\, d\gamma}{\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma} = \frac{2\pi\xi^2}{a^2(a^2 - \xi^2 - \eta^2)} + \frac{\pi}{a^2} \tag{3.29}$$

also

$$M_y = \frac{8\rho c}{3\pi}\iint_S \sqrt{a^2 - \xi^2 - \eta^2}\left\{-\frac{3}{2} f_0 - \mathrm{Re}\left[e^{-i\omega t}\left(\frac{3}{2} f - ik\xi\, f\right)\right] + \right.$$

$$\left.\frac{a^2}{2\pi}\left[f_0 + \mathrm{Re}(e^{-i\omega t}\, f)\right]\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi}\frac{\cos^2\gamma\, d\gamma}{\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma}\right\} d\xi\, d\eta \tag{3.30}$$

The value can now be computed for the frontal resistance W, which is composed of two parts. First, the normal force $(p_- - p_+)\, dx\, dy$ acting on an element of the wing $dx\, dy$ will have a component in the direction of the x-axis:

$$(p_- - p_+)\frac{\partial z}{\partial x}\,dx\,dy = (p_- - p_+)\left[\frac{\partial \zeta_0}{\partial x} + \mathrm{Re}\left(\frac{\partial \zeta}{\partial x}\,e^{-i\omega t}\right)\right]dx\,dy$$

if

$$z(x,y,t) = \zeta_0(x,y) + \mathrm{Re}\left[\zeta(x,y)\,e^{-i\omega t}\right]$$

is the equation of the surface of the wing. Integration of this expression gives the first part of the frontal resistance in the form:

$$W_1 = \iint_S (p_- - p_+)\left[\frac{\partial \zeta_0}{\partial x} + \mathrm{Re}\left(\frac{\partial \zeta}{\partial x}\,e^{-i\omega t}\right)\right]dx\,dy$$

$$= -2\rho c\iint_S\left\{\frac{\partial \varphi_0}{\partial x} + \mathrm{Re}\left[\left(\frac{\partial \Phi}{\partial x} + i\,k\,\Phi\right)e^{-i\omega t}\right]\right\}\left\{\frac{\partial \zeta_0}{\partial x} + \mathrm{Re}\left(\frac{\partial \zeta}{\partial x}\,e^{-i\omega t}\right)\right\}dx\,dy \tag{3.31}$$

In fact, the frontal resistance W will be less than W_1, since a suction force W_2 appears because of the presence of the sharp leading edge of the wing DAB; therefore,

$$W = W_1 - W_2 \tag{3.32}$$

The suction force W_2 is connected with the presence of a strong rarefaction near the edge of the wing. This rarefaction is taken into account principally by the square terms of the fundamental formulas (1.3) or (1.5) for the pressure and it is therefore unnecessary to employ these formulas here.

The suction force W_2 is computed from the law of conservation of momentum applied to a thin filament-like close region τ containing the forward semicircumference DAB of the circle S; region τ is bounded outside by surface σ and inside by part S' of the upper side of circle S adjacent to the semicircumference DAB and the part S'' of the lower side of the circle S. Figure 1 shows a section of these surfaces obtained by a passing plane through the z-axis.

The equation expressing the momentum law is projected on the x-axis:

$$-W_2 - \iint_\sigma p\cos(n,x)\,dS = \iiint_\tau \rho\,\frac{\partial v_x}{\partial t}\,d\tau + \rho\iint_\sigma v_n v_x\,dS + \rho\iint_{S'+S''} v_n v_x\,dS \tag{3.33}$$

The left-hand side is the sum of the projections on the x-axis of all the forces acting on the volume of fluid considered, and on the right-hand side is the total derivative with respect to time of the component on the x-axis of the momentum of this volume; this derivative consists of two parts, a volume integral connected with the local change of velocity and a surface integral expressing the transfer of the momentum of the particles of the fluid through the bounding surfaces of the volume τ.

Equation (3.33) may be written both for the stationary system of coordinates $O_1x_1y_1z_1$ and for the moving system of coordinates Oxyz.

For the stationary system of coordinates, expression (1.3) is used for the quantity p; moreover,

$$v_x = \frac{\partial\varphi}{\partial x} \qquad v_n = \frac{\partial\varphi}{\partial n} \tag{3.34}$$

By the theorem of Gauss

$$\iiint_{\tau} \rho \frac{\partial v_x}{\partial t_1}\, d\tau = \iiint_{\tau} \rho \frac{\partial^2\varphi}{\partial x\, \partial t_1}\, d\tau$$

$$= \iint_{\sigma} \rho \frac{\partial\varphi}{\partial t_1} \cos(n,x)\, dS + \iint_{S'+S''} \rho \frac{\partial\varphi}{\partial t_1} \cos(n,x)\, dS \tag{3.35}$$

From equation (1.3) and the equation just derived, the following expression is obtained from equation (3.33) after a number of simple transformations:

$$W_2 = \frac{\rho}{2} \iint_{\sigma} \left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right] \cos(n,x)\, dS - \rho \iint_{\sigma} \frac{\partial\varphi}{\partial x} \frac{\partial\varphi}{\partial n}\, dS -$$

$$\rho \iint_{S'+S''} \frac{\partial\varphi}{\partial t_1} \cos(n,x)\, dS - \rho \iint_{S'+S''} \frac{\partial\varphi}{\partial x} \frac{\partial\varphi}{\partial n}\, dS \tag{3.36}$$

Since $\partial\varphi/\partial t_1$ and $\partial\varphi/\partial x$ near the leading edge of the wing are of the order $\delta^{-1/2}$ and $\partial\varphi/\partial n$ and $\cos(n,x)$ are finite on the surface of the wing, the last integrals drop out when region τ is extended to the line DAB. The following limiting equation is therefore applicable:

$$W_2 = \lim_{\tau \to 0}\left\{\left[\frac{\rho}{2}\iint_\sigma\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right]\cos(n,x)\,dS - \rho\iint_\sigma \frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial n}\,dS\right]\right\} \tag{3.37}$$

For computation of the suction force W_2, the expressions must be found for the components of the velocity near the leading edge of the wing DAB. The velocity of the fluid particles near the leading edge of the wing are shown to be of the order of $\delta^{-1/2}$ if δ is the distance of the particle to the contour C of the circle S. From equations (1.15) and (2.7) it is evident that

$$\varphi(x,y,z,t) = \frac{1}{2\pi}\iint_S\left\{f_0(\xi,\eta) + \mathrm{Re}\left[f(\xi,\eta)e^{-i\omega t}\right]\right\}K(x,y,z,\xi,\eta)\,d\xi\,d\eta + \chi(x,y,z,t) \tag{3.38}$$

where the function $\chi(x,y,z,t)$ and its derivatives remain finite near the leading edge.

The behavior of the function is now examined more closely

$$U(x,y,z) = \iint_S f(\xi,\eta)\,K(x,y,z,\xi,\eta)\,d\xi\,d\eta \tag{3.39}$$

near the contour C of the circle S. Therefore,

$$\frac{\partial U}{\partial x} = \iint_S \frac{\partial K}{\partial x} f(\xi,\eta)\,d\xi\,d\eta$$

Since on C the function K becomes zero, the following equation results

$$\iint_S \frac{\partial K}{\partial \xi} f(\xi,\eta)\,d\xi\,d\eta = -\iint_S K\frac{\partial f}{\partial \xi}\,d\xi\,d\eta$$

showing the finiteness of this integral. Therefore

$$\frac{\partial U}{\partial x} = \iint_S \left(\frac{\partial K}{\partial x} + \frac{\partial K}{\partial \xi}\right) f(\xi,\eta)\, d\xi\, d\eta + O(1)$$

where $O(1)$ denotes a magnitude which remains finite when δ approaches 0. Put

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$$\frac{\partial K}{\partial x} + \frac{\partial K}{\partial \xi} = - \frac{2\sqrt{2}\, a \sqrt{a^2 - x^2 - y^2 - z^2 + R}}{\pi\left\{2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)\right\}} \times$$

$$\left\{\frac{x\sqrt{a^2 - \xi^2 - \eta^2}}{R} + \frac{\xi}{\sqrt{a^2 - \xi^2 - \eta^2}}\right\}$$

hence

$$\frac{\partial U}{\partial x} = - \frac{2\sqrt{2}}{\pi}\, a\sqrt{a^2-x^2-y^2-z^2+R} \iint_S \frac{f(\xi,\eta)}{2a^2r^2 + (a^2-\xi^2-\eta^2)(a^2-x^2-y^2-z^2+R)} \times$$

$$\left[\frac{x\sqrt{a^2-\xi^2-\eta^2}}{R} + \frac{\xi}{\sqrt{a^2-\xi^2-\eta^2}}\right] d\xi\, d\eta + O(1) \qquad (3.40)$$

The coordinates δ, θ, and α are introduced

$$x = (a + \delta \cos \alpha) \cos \theta \qquad y = (a + \delta \cos \alpha) \sin \theta \qquad z = \delta \sin \alpha \qquad (3.41)$$

Then

$$a^2 - x^2 - y^2 - z^2 = - 2a\delta \cos \alpha - \delta^2$$

$$R = \delta\sqrt{4a^2 + 4a\delta \cos \alpha + \delta^2} = 2a\delta + \ldots$$

$$\sqrt{a^2 - x^2 - y^2 - z^2 + R} = 2 \sin \frac{\alpha}{2} \sqrt{a\delta} + \ldots \qquad (3.42)$$

$$\sqrt{R - a^2 + x^2 + y^2 + z^2} = 2 \cos \frac{\alpha}{2} \sqrt{a\delta} + \ldots$$

The point with coordinates (x,y,z) is brought into correspondence with the point of the circumference C with the coordinates

$$x_0 = a \cos \theta \qquad y_0 = a \sin \theta \qquad z_0 = 0$$

and

$$r_0^2 = (x_0 - \xi)^2 + (y_0 - \eta)^2 = \xi^2 + \eta^2 + a^2 - 2a\xi \cos \theta - 2a\eta \sin \theta \tag{3.43}$$

Near the contour C, the principal part of the integral

$$J_1(x,y,z) = \iint_S \frac{2a^2 \sqrt{a^2 - \xi^2 - \eta^2}\, f(\xi,\eta)\, d\xi\, d\eta}{2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)} \tag{3.44}$$

is

$$N(\theta) = \iint_S f(\xi,\eta) \sqrt{a^2 - \xi^2 - \eta^2}\, \frac{d\xi\, d\eta}{r_0^2}$$

$$= \iint_S \frac{f(\xi,\eta) \sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta}{\xi^2 + \eta^2 + a^2 - 2a\xi \cos \theta - 2a\eta \sin \theta} \tag{3.45}$$

For this purpose, the following difference is estimated:

$$\Delta = J_1(x,y,z) - N(\theta)$$

The circle S is divided into two parts: the circle S_1 of radius $a - \varepsilon$; and the ring S_2 lying between the circumferences of radii $a - \varepsilon$ and a.

$$\Delta_1 = \iint_{S_1} f(\xi,\eta) \sqrt{a^2 - \xi^2 - \eta^2} \left\{ \frac{2a^2}{2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)} - \frac{1}{r_0^2} \right\} d\xi\, d\eta$$

$$= \iint_{S_1} f(\xi,\eta) \sqrt{a^2 - \xi^2 - \eta^2}\, \frac{2a^2r_0^2 - 2a^2r^2 - (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)}{r_0^2 \left\{ 2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R) \right\}}\, d\xi\, d\eta$$

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$$r_0^2 - r^2 = \xi^2 + \eta^2 + a^2 - 2a\xi \cos\theta - 2a\eta \sin\theta - (x-\xi)^2 - (y-\eta)^2 - z^2$$

$$= 2\delta \cos\alpha(\xi \cos\theta + \eta \sin\theta - a) - \delta^2$$

$$2a^2(r_0^2 - r^2) - (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)$$

$$= - 2a\delta r_0^2 \cos\alpha - (a^2 + \xi^2 + \eta^2)\,\delta^2 - (a^2 - \xi^2 - \eta^2)\,R$$

Since

$$r_0^2 \leqslant 4a^2 \qquad R \leqslant 2a\delta + \delta^2$$

therefore

$$\left|2a^2(r_0^2 - r^2) - (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)\right| \leqslant 2a\delta r_0^2 + 2a^2\delta^2 + (a^2 - \xi^2 - \eta^2)\,R$$

Hence if $|f(\xi,\eta)| < M$ in the circle S then

$$|\Delta_1| \leqslant 2a\delta M \iint_{S_1} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta}{2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)}$$

$$2a^2\delta^2 M \iint_{S_1} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta}{r_0^2\left[2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)\right]} +$$

$$RM \iint_{S_1} \frac{\sqrt{(a^2 - \xi^2 - \eta^2)^3}\, d\xi\, d\eta}{r_0^2\left[2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)\right]}$$

But by equation (2.24) of reference 1

$$\iint_{S} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta}{2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)} \leqslant \frac{\pi}{a}$$

Since

$$2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)$$

$$= 2a^2r_0^2 + 2a\delta r_0^2 \cos\alpha + (a^2 + \xi^2 + \eta^2)\,\delta^2 + (a^2 - \xi^2 - \eta^2)\,R$$

hence for $\delta < a/2$

$$2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R) \geqslant a^2r_0^2 \qquad (3.46)$$

and

$$\iint_{S_1} \frac{a^2\sqrt{a^2-\xi^2-\eta^2}\, d\xi\, d\eta}{r_0^2\left[2a^2r^2+(a^2-\xi^2-\eta^2)(a^2-x^2-y^2-z^2+R)\right]} \leqslant \iint_{S_1} \frac{1}{r_0^4}\sqrt{a^2-\xi^2-\eta^2}\, d\xi\, d\eta$$

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The last integral evidently does not depend on θ; hence it may be assumed that $\theta = 0$ and therefore

$$\iint_{S_1} \frac{1}{r_0^4}\sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta = \int_0^{2\pi}\int_0^{a-r} \frac{\sqrt{a^2 - \rho^2}\,\rho\, d\rho\, d\vartheta}{(\rho^2 - 2a\rho\cos\vartheta + a^2)^2} = \int_0^{a-\varepsilon} \frac{2\pi(a^2 + \rho^2)\rho\, d\rho}{\sqrt{(a^2 - \rho^2)^5}}$$

$$= \left[\frac{4\pi a^2}{3\sqrt{(a^2 - \rho^2)^3}} - \frac{2\pi}{\sqrt{a^2 - \rho^2}}\right]_{\rho=0}^{\rho=a-\varepsilon} = \frac{4\pi a^2}{3\sqrt{(2a\varepsilon - \varepsilon^2)^3}} - \frac{2\pi}{\sqrt{2a\varepsilon - \varepsilon^2}} + \frac{2\pi}{3a} < \frac{4\pi\sqrt{a}}{3\sqrt{\varepsilon^3}}$$

Similarly

$$\iint_{S_1} \frac{\sqrt{(a^2 - \xi^2 - \eta^2)^3}\, d\xi\, d\eta}{r_0^2\left[2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)\right]} \leqslant \frac{1}{a^2}\iint_{S_1} \frac{1}{r_0^4}\sqrt{(a^2 - \xi^2 - \eta^2)^3}\, d\xi\, d\eta$$

and

$$\iint_{S_1} \frac{1}{r_0^4}\sqrt{(a^2 - \xi^2 - \eta^2)^3}\, d\xi\, d\eta = \int_0^{2\pi}\int_0^{a-\varepsilon} \frac{\sqrt{(a^2 - \rho^2)^3}\,\rho\, d\rho\, d\vartheta}{(a^2 - 2a\rho\cos\vartheta + \rho^2)^2} = \int_0^{a-\varepsilon} \frac{2\pi(a^2 + \rho^2)\,\rho\, d\rho}{\sqrt{(a^2 - \rho^2)^3}}$$

$$= \left[\frac{4\pi a^2}{\sqrt{a^2 - \rho^2}} + 2\pi\sqrt{a^2 - \rho^2}\right]_{\rho=0}^{\rho=a-\varepsilon} = \frac{4\pi a^2}{\sqrt{2a\varepsilon - \varepsilon^2}} + 2\pi\sqrt{2a\varepsilon - \varepsilon^2} - 6\pi a < \frac{4\pi\sqrt{a^3}}{\sqrt{\varepsilon}}$$

As a result, the following inequality is obtained

$$|\Delta_1| \leqslant 2\pi M\delta + \frac{8\pi M\delta^2\sqrt{a}}{3\sqrt{\varepsilon^3}} + \frac{4\pi M(2a\delta + \delta^2)}{\sqrt{a\varepsilon}}$$

The difference is estimated

$$|\Delta_2| = \iint_{S_2} \frac{2a^2 f(\xi,\eta)\sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta}{2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)} - \iint_{S_2} \frac{1}{r_0^2} f(\xi,\eta) \sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta$$

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On account of equation (3.46)

$$|\Delta_2| \leq 3M \iint_{S_2} \frac{1}{r_0^2} \sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta$$

But

$$\iint_{S_2} \frac{1}{r_0^2} \sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta = \int_0^{2\pi}\int_{a-\varepsilon}^{a} \frac{\sqrt{a^2 - \rho^2}\,\rho\, d\rho\, d\phi}{\rho^2 - 2a\rho\cos\phi + a^2} = 2\pi\int_{a-\varepsilon}^{a} \frac{\rho\, d\rho}{\sqrt{a^2 - \rho^2}} = 2\pi\sqrt{2a\varepsilon - \varepsilon^2}$$

and therefore

$$|\Delta_2| < 6\pi M\sqrt{2a\varepsilon}$$

Thus for

$$\Delta = \Delta_1 + \Delta_2$$

the estimate is obtained

$$|\Delta| < 2\pi M\left\{\delta + \frac{4}{3}\,\frac{\delta^2\sqrt{a}}{\sqrt{\varepsilon^3}} + 4\delta\sqrt{\frac{a}{\varepsilon}} + \frac{2\delta^2}{\sqrt{a\varepsilon}} + 3\sqrt{2a\varepsilon}\right\}$$

Assuming

$$\varepsilon = \delta$$

yields

$$|\Delta| < 24\pi M\sqrt{a\delta}$$

Thus

$$\iint_S \frac{2a^2\sqrt{a^2 - \xi^2 - \eta^2}\, f(\xi,\eta)\, d\xi\, d\eta}{2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)} = N(\theta) + O\left(\sqrt{\delta}\right) \tag{3.47}$$

where $O(\alpha)$ denotes a magnitude, whose ratio to α remains finite when δ approaches zero.

An estimate of the second integral entering equation (3.40) is given:.

$$J_2(x,y,z) = \iint_S \frac{\xi f(\xi,\eta)\, d\xi\, d\eta}{\sqrt{a^2-\xi^2-\eta^2}\left[2a^2r^2+(a^2-\xi^2-\eta^2)(a^2-x^2-y^2-z^2+R)\right]} \tag{3.48}$$

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Again assuming $\delta < \alpha/2$ yields

$$|J_2| \leq \frac{M}{a}\iint_S \frac{d\xi\, d\eta}{\sqrt{a^2-\xi^2-\eta^2}\,(r_0^2+\delta^2)} = \frac{M}{2}\int_C^{a}\int_0^{2\pi} \frac{\rho\, d\rho\, d\phi}{\sqrt{a^2-\rho^2}\,(a^2-2a\rho\cos\phi+\rho^2+\delta^2)}$$

$$= \frac{2\pi M}{a}\int_0^a \frac{\rho\, d\rho}{\sqrt{a^2-\rho^2}\sqrt{(a^2+\rho^2+\delta^2)^2-4a^2\rho^2}}$$

but

$$\int_0^{a-\varepsilon} \frac{\rho\, d\rho}{\sqrt{a^2-\rho^2}\sqrt{(a^2+\rho^2+\delta^2)^2-4a^2\rho^2}} < \int_0^{a-\varepsilon} \frac{\rho\, d\rho}{\sqrt{(a^2-\rho^2)^3}}$$

$$= \left[\frac{1}{\sqrt{a^2-\rho^2}}\right]_{\rho=0}^{\rho=a-\varepsilon} = \frac{1}{\sqrt{2a\varepsilon-\varepsilon^2}} - \frac{1}{a} < \frac{1}{\sqrt{a\varepsilon}}$$

$$\int_{a-\varepsilon}^{a} \frac{\rho\, d\rho}{\sqrt{a^2-\rho^2}\sqrt{(a^2+\rho^2+\delta^2)^2-4a^2\rho^2}} < \int_{a-\varepsilon}^{a} \frac{\rho\, d\rho}{2a\delta\sqrt{a^2-\rho^2}}$$

$$= \left[-\frac{\sqrt{a^2-\rho^2}}{2a\delta}\right]_{\rho=a-\varepsilon}^{\rho=a} = \frac{\sqrt{2a\varepsilon-\varepsilon^2}}{2a\delta} < \frac{\sqrt{\varepsilon}}{\delta\sqrt{2a}}$$

hence

$$|J_2| < \frac{2\pi M}{a\sqrt{a}}\left(\frac{1}{\sqrt{\varepsilon}} + \frac{\sqrt{\varepsilon}}{\delta}\right)$$

and for $\varepsilon = \delta$

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$$|J_2| < \frac{4\pi M}{a\sqrt{a\delta}} \tag{3.49}$$

From equation (3.40) and equation (3.42), the following is obtained on account of the estimates (3.47) and (3.49):

$$\frac{\partial U}{\partial x} = -\frac{\sqrt{2}\,N(\theta)\,x\sqrt{a^2 - x^2 - y^2 - z^2 + R}}{\pi a R} + O(1) \tag{3.50}$$

In exactly the same way, there is obtained

$$\frac{\partial U}{\partial y} = -\frac{\sqrt{2}\,N(\theta)\,y\sqrt{a^2 - x^2 - y^2 - z^2 + R}}{\pi a R} + O(1) \tag{3.51}$$

Finally,

$$\frac{\partial U}{\partial z} = \iint_S \frac{\partial K}{\partial z} f(\xi,\eta)\,d\xi\,d\eta$$

But

$$\frac{\partial K}{\partial z} = -\frac{2z}{\pi r^3}\arctan A + \frac{2A}{\pi(1 + A^2)}\left[-\frac{z}{r^3} + \frac{z(a^2 + x^2 + y^2 + z^2 - R)}{rR(a^2 - x^2 - y^2 - z^2 + R)}\right]$$

where

$$A = \frac{\sqrt{a^2 - \xi^2 - \eta^2}\sqrt{a^2 - x^2 - y^2 - z^2 + R}}{ar\sqrt{2}}$$

Assuming $z > 0$,

$$\iint_S \frac{z}{r^3}\,d\xi\,d\eta \leq 2\pi$$

hence

$$\left| \iint_S \left(\frac{2z}{\pi r^3} \text{arc tan } A + \frac{2}{\pi} \frac{A}{1 + A^2} \frac{z}{r^3} \right) f(\xi,\eta) \, d\xi \, d\eta \right| \leq 2(\pi + 1) M$$

and therefore

$$\frac{\partial U}{\partial z} = \frac{2\sqrt{2}\, az(a^2 + x^2 + y^2 + z^2 - R)}{\pi R \sqrt{a^2 - x^2 - y^2 - z^2 + R}} \iint_S \frac{f(\xi,\eta) \sqrt{a^2 - \xi^2 - \eta^2} \, d\xi \, d\eta}{2z^2 r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)} + O(1)$$

Again use is made of equations (3.47) and (3.42) and the fact that for $z > 0$

$$\frac{z}{\sqrt{a^2 - x^2 - y^2 - z^2 + R}} = \frac{1}{2a} \sqrt{R - a^2 + x^2 + y^2 + z^2}$$

without difficulty:

$$\frac{\partial U}{\partial z} = \frac{a^2 + x^2 + y^2 + z^2 - R}{\pi R a^2 \sqrt{2}} N(\theta) \sqrt{R - a^2 + x^2 + y^2 + z^2} + O(1) \tag{3.52}$$

From what has been said previously about equation (3.38) it is evident that if

$$F(\xi,\eta,t) = f_0(\xi,\eta) + f_1(\xi,\eta) \cos \omega t + f_2(\xi,\eta) \sin \omega t \tag{3.53}$$

$$N(\theta,t) = \iint_S \frac{F(\xi,\eta,t) \sqrt{a^2 - \xi^2 - \eta^2} \, d\xi \, d\eta}{\xi^2 + \eta^2 + a^2 - 2a\xi \cos \theta - 2a\eta \sin \theta} \tag{3.54}$$

the following results

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$$\left.\begin{aligned}
\frac{\partial\varphi}{\partial x} &= -\frac{x\sqrt{a^2 - x^2 - y^2 - z^2 + R}}{\sqrt{2}\,\pi^2 aR} N(\theta,t) + O(1) \\
\frac{\partial\varphi}{\partial y} &= -\frac{y\sqrt{a^2 - x^2 - y^2 - z^2 + R}}{\sqrt{2}\,\pi^2 aR} N(\theta,t) + O(1) \\
\frac{\partial\varphi}{\partial z} &= \frac{1}{2\sqrt{2}\,\pi^2 a^2 R}\left(a^2 + x^2 + y^2 + z^2 - R\right) \times \\
&\quad \sqrt{R - a^2 + x^2 + y^2 + z^2}\, N(\theta,t) + O(1)
\end{aligned}\right\} \tag{3.55}$$

or, in the coordinates δ, θ, α

$$\left.\begin{aligned}
\frac{\partial\varphi}{\partial x} &= -\frac{N(\theta,t)\cos\theta\sin\left(\frac{1}{2}\alpha\right)}{\pi^2\sqrt{2a\delta}} + O(1) \\
\frac{\partial\varphi}{\partial y} &= -\frac{N(\theta,t)\sin\theta\sin\left(\frac{1}{2}\alpha\right)}{\pi^2\sqrt{2a\delta}} + O(1) \\
\frac{\partial\varphi}{\partial z} &= \frac{N(\theta,t)\cos\left(\frac{1}{2}\alpha\right)}{\pi^2\sqrt{2a\delta}} + O(1)
\end{aligned}\right\} \tag{3.56}$$

The computation of the suction force W_2 by equation (3.37) is considered. An arc D'AB' of the circumference C is taken symmetrical with respect to the x-axis with subtending angle $2\theta_0 < \pi$. For the surface σ, the part σ_0 is taken of the surface determined by equations (3.41) for constant δ_0, where θ changes from $-\theta_0$ to $+\theta_0$ and α from $-\pi$ to $+\pi$ and two bases; one of which, σ_1, corresponds to $\theta = \theta_0$ and the other, σ_2, corresponds to $\theta = -\theta_0$, where on these bases δ varies from 0 to δ_0 and α from $-\pi$ to $+\pi$.

On the toroidal surface:

$$\cos(n,x) = \cos\alpha\cos\theta \qquad \cos(n,y) = \cos\alpha\sin\theta \qquad \cos(n,z) = \sin\alpha$$

$$\frac{\partial\varphi}{\partial n} = \frac{\partial\varphi}{\partial x}\cos(n,x) + \frac{\partial\varphi}{\partial y}\cos(n,y) + \frac{\partial\varphi}{\partial z}\cos(n,z) = \frac{N(\theta,t)\sin\left(\frac{1}{2}\alpha\right)}{\pi^2\sqrt{2a\delta}} + O(1)$$

Hence simple computation shows that

$$\frac{1}{2}\iint_{\sigma_0}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right]\cos(n,x)\, dS - \iint_{\sigma_0}\frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial n}\, dS$$

$$= \frac{1}{2}\int_{-\theta_0}^{\theta_0}\int_{-\pi}^{\pi}\frac{1}{2\pi^4}N^2(\theta,t)\cos\alpha\cos\theta\, d\theta\, d\alpha + \int_{-\theta_0}^{\theta_0}\int_{-\pi}^{\pi}\frac{1}{2\pi^4}N^2(\theta,t)\cos\theta\sin^2\frac{1}{2}\alpha\, d\theta\, d\alpha +$$

$$O\left(\sqrt{\delta_0}\right) = \frac{1}{2\pi^3}\int_{-\theta_0}^{\theta_0}N^2(\theta,t)\cos\theta\, d\theta + O\left(\sqrt{\delta_0}\right)$$

In the same manner, the integrals taken over the bases σ_1 and σ_2 have the order $O(\delta_0)$. Hence if δ_0 approaches zero, for the suction force developed along the arc D'AB', the following expression is obtained

$$\frac{\rho}{2\pi^3}\int_{-\theta_0}^{\theta_0}N^2(\theta,t)\cos\theta\, d\theta$$

Now when θ_0 approaches $\pi/2$, the required expression for the suction force W_2 is obtained in the following form:

$$W_2 = \frac{\rho}{2\pi^3}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}N^2(\theta,t)\cos\theta\, d\theta \tag{3.57}$$

The mean value of the frontal resistance is found. Equation (3.31) shows that for the mean value of W_1

$$\overline{W}_1 = -2\rho c\iint_S\left[\frac{\partial\varphi_0}{\partial x}\frac{\partial\zeta_0}{\partial x} + \frac{1}{2}\left(\frac{\partial\varphi_1}{\partial x} - k\varphi_2\right)\frac{\partial\zeta_1}{\partial x} + \frac{1}{2}\left(\frac{\partial\varphi_2}{\partial x} + k\varphi_1\right)\frac{\partial\zeta_2}{\partial x}\right]dx\, dy \tag{3.58}$$

In the same way, for the mean value of the suction force

$$\overline{W}_2 = \frac{\rho}{2\pi^3}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\left[N_0^2(\theta) + \frac{1}{2}N_1^2(\theta) + \frac{1}{2}N_2^2(\theta)\right]\cos\theta\, d\theta \tag{3.59}$$

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where

$$N_k(\theta) = \iint_S \frac{f_k(\xi,\eta)\sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta}{\xi^2 + \eta^2 + a^2 - 2a\xi\cos\theta - 2a\eta\sin\theta} \qquad (k = 0,1,2) \tag{3.60}$$

For the mean value of the frontal resistance

$$\overline{W} = \overline{W}_1 - \overline{W}_2 \tag{3.61}$$

4. Example

If a plane wing varies its angle of attack periodically according to the harmonic law so that the equation of its surface is

$$z = (\beta_0 + \beta_1 \cos\omega t)\, x \tag{4.1}$$

in the notation of section 1, the following is obtained

$$\zeta_0(x,y) = \beta_0 x \qquad \zeta_1(x,y) = \beta_1 x \qquad \zeta_2(x,y) = 0$$

and therefore

$$\begin{gathered} Z_0(x,y) = -c\beta_0 \qquad Z_1(x,y) = -c\beta_1 \qquad Z_2(x,y) = -ck\beta_1 x \\ Z(x,y) = Z_1 + iZ_2 = -c\beta_1(1 + i\,x) \end{gathered} \tag{4.2}$$

The function $f(x,y)$ corresponding to this value of the function $Z(x,y)$ is determined by equation (2.18) where $g(y)$ is the solution of integral equation (2.19).

Consideration is restricted to the solution of the inverse problem by assuming that

$$f_0(x,y) = A_0 \qquad f(x,y) = A + Bx$$

where A and B are constant complex numbers and A_0 is a constant real number and the shape of the wing is determined corresponding to this function. By such a method it is possible to obtain also an approximate solution of the direct problem of the nonsteady motion of a wing according to the law (4.1) for the case of small frequencies of vibration.

The forces acting on the wing are determined. For determination of the lift force P, use is made of equation (3.13). The following relations are used

$$\iint_S \sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta = \frac{2}{3}\pi a^3 \qquad \iint_S \xi\sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta = 0 \tag{4.3}$$

as are equations (3.5) and (3.26), yielding without difficulty

$$P = -\frac{4\rho c}{\pi}\left\{\mathrm{Re}\left(\frac{2}{3} i\pi k a^3 Ae^{-i\omega t}\right) + \right.$$

$$\left. a^2\left[A_0 + \mathrm{Re}(Ae^{-i\omega t})\right]\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \cos\gamma\, d\gamma + \frac{2}{3} a^3\, \mathrm{Re}(Be^{-i\omega t})\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \cos^2\gamma\, d\gamma\right\}$$

or

$$P = \frac{8\rho c a^2}{\pi} A_0 + \frac{8\rho c a^2}{\pi}\mathrm{Re}\left[Ae^{-i\omega t}\left(1 - \frac{ika}{3}\right)\right] - \frac{4\rho c a^3}{3}\mathrm{Re}(Be^{-i\omega t}) \tag{4.4}$$

The moment of the pressure forces about the x-axis equals zero on account of symmetry:

$$M_x = 0 \tag{4.5}$$

If the moment of the pressure forces about the y-axis is determined by equation (3.28) and, in addition to the previously mentioned formulas, use is made also of the formula

$$\iint_S \xi^2\sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta = \frac{2}{15}\pi a^5$$

$$M_y = \frac{8\rho c}{3\pi}\left\{\frac{2}{15}\pi a^5 \operatorname{Re}(ikBe^{-i\omega t}) - \right.$$

$$\left. a^3\left[A_0 + \operatorname{Re}(Ae^{-i\omega t})\right]\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\cos^2\gamma\, d\gamma - \frac{2}{3}a^4 \operatorname{Re}(Be^{-i\omega t})\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\cos^3\gamma\, d\gamma\right\}$$

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or

$$M_y = -\frac{4\rho ca^3}{3}A_0 - \frac{4\rho ca^3}{3}\operatorname{Re}(Ae^{-i\omega t}) - \frac{64\rho ca^4}{27\pi}\operatorname{Re}\left[Be^{-i\omega t}\left(1 - \frac{3\pi}{20}iak\right)\right] \tag{4.6}$$

The frontal resistance is computed. First the suction force is computed:

If

$$A = A_1 + iA_2 \qquad B = B_1 + iB_2$$

according to equation (3.53)

$$F(\xi,\eta,t) = A_0 + (A_1 + B_1\xi)\cos\omega t + (A_2 + B_2\xi)\sin\omega t$$

If equation (3.54) is applied and use is made of equations (3.5) and (3.26),

$$N(\theta,t) = 2\pi a(A_0 + A_1\cos\omega t + A_2\sin\omega t) + \frac{4}{3}\pi a^2\cos\theta\,(B_1\cos\omega t + B_2\sin\omega t)$$

Equation (3.57) yields without difficulty the expression for the suction force:

$$W_2 = \frac{\rho}{2\pi^3}\left\{8\pi^2 a^2(A_0 + A_1\cos\omega t + A_2\sin\omega t)^2 + \right.$$

$$\frac{8}{3}\pi^3 a^3(A_0 + A_1\cos\omega t + A_2\sin\omega t)(B_1\cos\omega t + B_2\sin\omega t) +$$

$$\left.\frac{64}{27}\pi^2 a^4(B_1\cos\omega t + B_2\sin\omega t)^2\right\}.$$

or

$$W_2 = \frac{4\rho a^2}{\pi}\left\{A_0^2 + \frac{1}{2}A_1^2 + \frac{1}{2}A_2^2 + \frac{\pi}{6}aA_1B_1 + \frac{\pi}{6}aA_2B_2 + \frac{4}{27}a^2B_1^2 + \frac{4}{27}a^2B_2^2 + \right.$$

$$\left(2A_0A_1 + \frac{\pi}{3}aA_0B_1\right)\cos\omega t + \left(2A_0A_2 + \frac{\pi}{3}aA_0B_2\right)\sin\omega t +$$

$$\left(\frac{1}{2}A_1^2 - \frac{1}{2}A_2^2 + \frac{\pi}{6}aA_1B_1 - \frac{\pi}{6}aA_2B_2 + \frac{4}{27}a^2B_1^2 - \frac{4}{27}a^2B_2^2\right)\cos 2\omega t +$$

$$\left.\left(A_1A_2 + \frac{\pi}{6}aA_1B_2 + \frac{\pi}{6}aA_2B_1 + \frac{8}{27}a^2B_1B_2\right)\sin 2\omega t\right\} \tag{4.7}$$

The total frontal resistance is obtained by the equation

$$W = W_1 - W_2$$

where W_1 is determined by equation (3.31)

$$W_1 = -2\rho c\iint_S\left\{\frac{\partial\varphi_0}{\partial x} + \mathrm{Re}\left[\left(\frac{\partial\Phi}{\partial x} + ik\Phi\right)e^{-i\omega t}\right]\right\}\left\{\frac{\partial\zeta_0}{\partial x} + \mathrm{Re}\left(\frac{\partial\zeta}{\partial x}e^{-i\omega t}\right)\right\}dx\,dy \tag{4.8}$$

For the mean value of the frontal resistance the following is obtained:

$$\overline{W} = \overline{W}_1 - \overline{W}_2 \tag{4.9}$$

where

$$\overline{W}_2 = \frac{4\rho a^2}{\pi}\left(A_0^2 + \frac{1}{2}A_1^2 + \frac{1}{2}A_2^2 + \frac{\pi}{6}aA_1B_1 + \frac{\pi}{6}aA_2B_2 + \frac{4}{27}a^2B_1^2 + \frac{4}{27}a^2B_2^2\right) \tag{4.10}$$

$$\overline{W}_1 = -2\rho c\iint_S\left\{\frac{\partial\varphi_0}{\partial x}\frac{\partial\zeta_0}{\partial x} + \frac{1}{2}\mathrm{Re}\left[\left(\frac{\partial\Phi}{\partial x} + ik\Phi\right)\frac{\partial\zeta}{\partial x}\right]\right\}dx\,dy \tag{4.11}$$

For determination of the functions $\zeta_0(x,y)$ and $\zeta(x,y)$ characterizing the shape of the wing, equation (1.16) is used.

$$-c\frac{\partial\zeta_0}{\partial x} = Z_0(x,y) \qquad -c\left(\frac{\partial\zeta}{\partial x} + ik\zeta\right) = Z(x,y) \tag{4.12}$$

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where by equation (2.17) in this case

$$Z_0(x,y) = -A_0 + g_0(y) \qquad Z(x,y) = -A - Bx + g(y)e^{-ikx} \qquad (4.13)$$

and the functions $g_0(y)$ and $g(y)$ in this case according to equation (2.16) have the form:

$$g_0(y) = \frac{aA_0}{2\pi^3} \times$$

$$\iint_S \int_{+\infty}^{\sqrt{a^2-y^2}} \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{(x^2+y^2-a^2)^{-\frac{1}{2}}(a^2-\xi^2-\eta^2)^{\frac{1}{2}} \cos\gamma \, d\gamma \, dx \, d\xi \, d\eta}{(x^2+y^2+a^2-2ax\cos\gamma-2ay\sin\gamma)(\xi^2+\eta^2+a^2-2a\xi\cos\gamma-2a\eta\sin\gamma)}$$

$$g(y) = \frac{a}{2\pi^3} \times$$

$$\iint_S \int_{+\infty}^{\sqrt{a^2-y^2}} \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{e^{ikx}(x^2+y^2-a^2)^{-\frac{1}{2}}(a^2-\xi^2-\eta^2)^{\frac{1}{2}}(A+B\xi) \cos\gamma \, d\gamma \, dx \, d\xi \, d\eta}{(x^2+y^2+a^2-2ax\cos\gamma-2ay\sin\gamma)(\xi^2+\eta^2+a^2-2a\xi\cos\gamma-2a\eta\sin\gamma)}$$

Equations (3.5) and (3.26) yield

$$\left.\begin{aligned} g_0(y) &= \frac{a^2A_0}{\pi^2} \int_{+\infty}^{\sqrt{a^2-y^2}} \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{\cos\gamma \, d\gamma \, dx}{\sqrt{x^2+y^2-a^2}\,(x^2+y^2+a^2-2ax\cos\gamma-2ay\sin\gamma)} \\ g(y) &= \frac{a^2}{\pi^2} \int_{+\infty}^{\sqrt{a^2-y^2}} \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{e^{ikx}(A+2/3\,aB\cos\gamma)\cos\gamma \, d\gamma \, dx}{\sqrt{x^2+y^2-a^2}\,(x^2+y^2+a^2-2ax\cos\gamma-2ay\sin\gamma)} \end{aligned}\right\} \quad (4.14)$$

Integration of equations (4.12) yields

$$\zeta_0(x,y) = \frac{A_0}{c} x - \frac{1}{c} g_0(y)\, x + h_0(y)$$

$$\zeta(x,y) = \left(A + \frac{iB}{k}\right) \frac{1 - e^{-ikx}}{ikc} - \frac{iBx}{kc} - \frac{1}{c}\, g(y)\, xe^{-ikx} + h(y)\, e^{-ikx} \tag{4.15}$$

where $h_0(y)$ and $h(y)$ are arbitrary functions of y.

The function $g_0(y)$ was obtained in reference 1, where, however, errors slipped into the computations. Setting

$$y = -a\cos\theta \qquad H_0(\theta) = \frac{\pi^2}{A_0} \sin\theta\, g_0(-a\cos\theta) \qquad (0<\theta<\pi) \tag{4.16}$$

gives in place of equation (4.22) of reference 1

$$H_0(\theta) = \frac{\pi^2}{4} \sin\theta + \frac{1}{8} \sin\theta \left(\ln \frac{1 + \sin\frac{1}{2}\theta}{1 - \sin\frac{1}{2}\theta}\right)^2 + \frac{1}{8} \sin\theta \left(\ln \frac{1 - \cos\frac{1}{2}\theta}{1 + \cos\frac{1}{2}\theta}\right)^2 +$$

$$\cos\frac{1}{2}\theta \ln \frac{1 - \sin\frac{1}{2}\theta}{1 + \sin\frac{1}{2}\theta} + \sin\frac{1}{2}\theta \ln \frac{1 - \cos\frac{1}{2}\theta}{1 + \cos\frac{1}{2}\theta} \tag{4.17}$$

Hence setting $h_0(y) = 0$ and $A_0 = \alpha c$ in place of equation (4.23) of reference 1 yields

$$\zeta_0(x,y) = \alpha x\left\{\frac{3}{4} - \frac{1}{8\pi^2}\left(\ln \frac{\sqrt{2a} + \sqrt{a + y}}{\sqrt{2a} - \sqrt{a + y}}\right)^2 - \frac{1}{8\pi^2}\left(\ln \frac{\sqrt{2a} + \sqrt{a - y}}{\sqrt{2a} - \sqrt{a - y}}\right)^2 - \right.$$

$$\left. \frac{\sqrt{2a}}{2\pi^2 \sqrt{a + y}} \ln \frac{\sqrt{2a} - \sqrt{a + y}}{\sqrt{2a} + \sqrt{a + y}} - \frac{\sqrt{2a}}{2\pi^2 \sqrt{a - y}} \ln \frac{\sqrt{2a} - \sqrt{a - y}}{\sqrt{2a} + \sqrt{a + y}}\right\} \tag{4.18}$$

In particular for $y = 0$ and $y = \pm a/2$ the following values are obtained in place of those given in reference 1:

$$\zeta(x,0) = \alpha x\left[\frac{3}{4} - \frac{1}{\pi^2}\ln^2(\sqrt{2}+1) + \frac{2\sqrt{2}}{\pi^2}\ln(\sqrt{2}+1)\right] = 0.9263\ \alpha x$$

$$\zeta\left(x, \pm\frac{a}{2}\right) =$$

$$\alpha x\left[\frac{3}{4} - \frac{1}{2\pi^2}\ln^2(2+\sqrt{3}) - \frac{1}{8\pi^2}\ln^2 3 + \frac{2}{\pi^2\sqrt{3}}\ln(2+\sqrt{3}) + \frac{1}{\pi^2}\ln 3\right] = 0.9146\ \alpha x$$

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In the same way, the expansion given in reference 1 of the function $H_0(\theta)$ in a trigonometric series in the interval $0 \leq \theta \leq \pi$ should be replaced by the following:

$$H_0(\theta) = \sin\theta\left(\frac{\pi^2}{2} - 4\right) + \sum_{k=1}^{\infty}\frac{\sin(2k+1)\theta}{k(k+1)}\left(\frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{4k+1}\right) \tag{4.19}$$

that is,

$$H_0(\theta) = \sum_{k=0}^{\infty}\beta_{2k+1}\sin(2k+1)\theta$$

where

$$\beta_1 = 0.9348 \qquad \beta_5 = 0.1312 \qquad \beta_9 = 0.0504$$
$$\beta_3 = 0.2667 \qquad \beta_7 = 0.0796 \qquad \ldots\ldots$$

In connection with this, corrections should also be applied to the numerical values, which are given in reference 1, of the coefficients B_n of the trigonometric series for the circulation obtained by the usual theory

$$B_1 = 2.2125\ \alpha ca \qquad B_5 = -0.0296\ \alpha ca \qquad B_9 = -0.0067\ \alpha ca$$

$$B_3 = -0.0934\ \alpha ca \qquad B_7 = -0.0133\ \alpha ca \qquad \ldots\ldots$$

Hence for the lift force in place of equation (4.29) of reference 1, the following is obtained:

$$P_0 = \frac{1}{2}\pi\rho ca \qquad B_1 = 3.4755\ \rho c^2 a^2 \alpha$$

which exceeds the accurate value by 36 percent. For the induced drag, in place of equation (4.30) of reference 1, the following is obtained

$$W_0 = 1.9350\ \rho c^2 \alpha^2 a^2$$

which exceeds the accurate value by 87 percent.

Corrections are made in the third example given in reference 1. The value of the definite integral is:

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$$\int_0^1 \frac{\arctan y}{\sqrt{1 - y^2}}\, dy = \frac{\pi^2}{8} - \frac{1}{2} \ln^2(\sqrt{2} + 1)$$

Hence in equation (4.52) of reference 1 the coefficient of $\sin\theta \cos\theta$ is simplified and assumes the value $-3\pi^2/8$. In equation (4.53) the coefficient of $\sin 2\theta$ was incorrectly computed; its correct value is

$$\delta_2 = -\frac{3\pi^2}{8} + \frac{32}{9} = -0.14555$$

In this connection, the value of the coefficient B_2 should also be corrected:

$$B_2 = -0.7436\ \alpha c a^2$$

For the induced drag and the moment of the forces about the x-axis, in place of the values of equation (4.55) of reference 1, the following is obtained:

$$W = 0.4343\ \rho \alpha^2 c^2 a^4 \qquad M_x = 0.5840\ \rho \alpha^2 c^2 a^4$$

the first gives an error of 140 percent; the second of 55 percent.

The shape of the wing obtained

$$z(x,y,t) = \frac{A_0}{c} x - \frac{1}{c} g_0(y)\, x + \operatorname{Re}\left\{e^{-i\omega t}\left[\left(A + \frac{i\Gamma}{k}\right)\frac{1 - e^{-ikx}}{ikc} - \frac{iBx}{kc} - \frac{1}{c} g(y)\, x e^{-ikx}\right]\right\} \qquad (4.20)$$

depends on the frequency of the vibrations and is deformed during the vibrations. The rigid wing is of greater interest.

It is possible with the aid of the results obtained to obtain an approximate solution of the problem of the vibrations of a plane circular wing for small frequencies of vibration.

The case is now considered of a wing varying its angle of attack periodically according to the harmonic law (4.1), so that equation (4.2) holds.

If

$$f_0(x,y) = A_0 \qquad f(x,y) = A + Bx$$

equation (4.2) yields

$$Z_0(x,y) = -A_0 + g(y) \qquad Z(x,y) = -A - Bx + g(y)\, e^{-ikx} \qquad (4.21)$$

If

$$G_0(y) = \frac{a^2}{\pi^2} \int_{+\infty}^{\sqrt{a^2-y^2}} \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{\cos\gamma\, d\gamma\, dx}{\sqrt{x^2+y^2-a^2}\,\left(x^2+y^2+a^2-2ax\cos\gamma-2ay\sin\gamma\right)}$$

$$G_1(y) = \frac{a^2}{\pi^2} \int_{+\infty}^{\sqrt{a^2-y^2}} \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{e^{ikx}\cos\gamma\, d\gamma\, dx}{\sqrt{x^2+y^2-a^2}\,\left(x^2+y^2+a^2-2ax\cos\gamma-2ay\sin\gamma\right)}$$

$$G_2(y) = \frac{a^2}{\pi^2} \int_{+\infty}^{\sqrt{a^2-y^2}} \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{e^{ikx}\cos^2\gamma\, d\gamma\, dx}{\sqrt{x^2+y^2-a^2}\,\left(x^2+y^2+a^2-2ax\cos\gamma-2ay\sin\gamma\right)}$$

(4.22)

Then

$$g_0(y) = A_0G_0(y) \qquad g(y) = AG_1(y) + BG_2(y) \qquad (4.23)$$

In place of $G_k(y)$, their mean values are taken over the area of the wing:

$$\tilde{G}_k = \left\{\int_{-a}^{a} G_k(y)\sqrt{a^2-y^2}\,dy\right\} : \left\{\int_{-a}^{a} \sqrt{a^2-y^2}\,dy\right\} = \frac{2}{\pi a^2}\int_{-a}^{a} G_k(y)\sqrt{a^2-y^2}\,dy$$

$$(k = 0,1,2) \qquad (4.24)$$

If the frequency of the vibrations is assumed small, or more accurately, the magnitude ka is assumed small, the expansion

$$e^{-ikx} = 1 - ikx - \frac{1}{2} k^2x^2 - \ldots$$

may be limited to the first two terms.

From equation (4.21), the following approximate expressions were obtained

$$\begin{aligned} Z_0(x,y) &= -A_0 + A_0\widetilde{G}_0 \\ Z(x,y) &= -A - Bx + (1 - ikx)(A\widetilde{G}_1 + B\widetilde{G}_2) \end{aligned} \tag{4.25}$$

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Comparison with equation (4.2) results in:

$$-c\beta_0 = -A_0 + A_0\widetilde{G}_0$$

$$-c\beta_1 = -A + A\widetilde{G}_1 + B\widetilde{G}_2$$

$$-c\beta_1 ik = -B - ik(A\widetilde{G}_1 + B\widetilde{G}_2)$$

whence

$$A_0 = \frac{c\beta_0}{1 - \widetilde{G}_0} \qquad A = \frac{c\beta_1(1 + 2ik\widetilde{G}_2)}{1 - \widetilde{G}_1 + ik\widetilde{G}_2} \qquad B = \frac{c\beta_1 ik(1 - 2\widetilde{G}_1)}{1 - \widetilde{G}_1 + ik\widetilde{G}_2} \tag{4.26}$$

The following is computed

$$\widetilde{G}_0 = \frac{2}{\pi a^2}\int_{-a}^{a} G_0(y)\sqrt{a^2 - y^2}\, dy = \frac{2}{\pi}\int_0^{\pi} G_0(-a\cos\theta)\sin^2\theta\, d\theta$$

But by equation (4.16)

$$\sin\theta\, G_0(-a\cos\theta) = \sin\theta\,\frac{g_0(-a\cos\theta)}{A_0} = \frac{1}{\pi^2}H_0(\theta)$$

hence, expansion (4.19) is used, yielding

$$\tilde{G}_0 = \frac{2}{\pi^3}\int_0^{\pi} H_0(\theta) \sin \theta \, d\theta = \frac{2}{\pi^3}\left(\frac{\pi^2}{2} - 4\right)\frac{\pi}{2} = \frac{1}{2} - \frac{4}{\pi^2}$$

and therefore

$$\tilde{G}_0 = \frac{1}{2} - 0.4053 = 0.0947 \qquad A_0 = 1.105c\beta_0 \tag{4.27}$$

Equations (4.26) show that in computing $\tilde{G}_1$ it is sufficient to use the terms of first-order smallness relative to ka, while in computing $\tilde{G}_2$ it is sufficient to use the principal term not depending on k. For small ka the following results

$$\tilde{G}_1 = \tilde{G}_0 + ik\tilde{G}_{11} + O\left(k^2a^2 \ln \frac{1}{ka}\right) \qquad \tilde{G}_2 = \tilde{G}_{20} + O(ka^2) \tag{4.28}$$

where $\tilde{G}_{11}$ and $\tilde{G}_{20}$ are the mean values over the area of the circle S of the functions

$$G_{11}(y) = \frac{a^2}{\pi^2}\int_{\infty}^{\sqrt{a^2-y^2}}\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{x \cos \gamma \, d\gamma \, dx}{\sqrt{x^2 + y^2 - a^2}\,(x^2 + y^2 + a^2 - 2ax \cos \gamma - 2ay \sin \gamma)} \tag{4.29}$$

$$G_{20}(y) = \frac{2}{3}\frac{a^3}{\pi^2}\int_{+\infty}^{\sqrt{a^2-y^2}}\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{\cos^2 \gamma \, d\gamma \, dx}{\sqrt{x^2 + y^2 - a^2}\,(x^2 + y^2 + a^2 - 2ax \cos \gamma - 2ay \sin \gamma)} \tag{4.30}$$

In fact,

$$\tilde{G}_1 - \tilde{G}_0 - ik\tilde{G}_{11} = \frac{2}{\pi^3}\int_{-a}^{a}\int_{\infty}^{\sqrt{a^2-y^2}}\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} G^*(x,y,\gamma)\, d\gamma \, dx \, dy \tag{4.31}$$

where

$$G^*(x,y,\gamma) = \frac{(e^{ikx} - 1 - ikx) \cos \gamma \sqrt{a^2 - y^2}}{\sqrt{x^2 + y^2 - a^2}\,(x^2 + y^2 + a^2 - 2ax \cos \gamma - 2ay \sin \gamma)}$$

The interval of integration with respect to x is divided into two parts: from $\sqrt{a^2 - y^2}$ to $2a$ and from $2a$ to ∞. Since for $\alpha > 0$

$$\left|e^{i\alpha} - 1 - i\alpha\right| < \alpha^2$$

in the interval $\sqrt{a^2 - y^2} \leq x \leq 2a$, $\left|e^{ikx} - 1 - ikx\right| \leq (2ka)^2$ and therefore

$$\left|\frac{2}{\pi^3}\int_{-a}^{a}\int_{2a}^{\sqrt{a^2-y^2}}\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} G^*(x,y,\gamma)\,d\gamma\,dx\,dy\right| < (2ka)^2\,\tilde{G}_0 < 0.38k^2a^2 \tag{4.32}$$

On the other hand, for $x \geq 2a$, $|y| < a$, $\pi/2 \leqslant \gamma \leqslant 3\pi/2$ the inequality holds

$$x^2 + y^2 - a^2 > \frac{3}{4}x^2 \qquad (x - a\cos\gamma)^2 + (y - a\sin\gamma)^2 > x^2$$

As

$$\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \cos\gamma\,d\gamma = -2$$

$$\int_{-a}^{a}\sqrt{a^2 - y^2}\,dy = \frac{\pi a^2}{2} \quad e^{ikx} - 1 - ikx = \cos kx - 1 + i(\sin kx - kx)$$

the following inequalities are obtained when, for clarity, ka is assumed < 1,

$$\left|\frac{2}{\pi^3}\int_{-a}^{a}\int_{\infty}^{2a}\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} G^*(x,y,\gamma)\,d\gamma\,dx\,dy\right| < \frac{4a^2}{\sqrt{3}\,\pi^2}\left\{\int_{2a}^{\infty}\frac{1 - \cos kx}{x^3}\,dx + \int_{2a}^{\infty}\frac{kx - \sin kx}{x^3}\,dx\right\}$$

$$= \frac{4a^2k^2}{\sqrt{3}\,\pi^2}\left\{\int_{2ak}^{\infty}\frac{1 - \cos u}{u^3}\,du + \int_{0}^{\infty}\frac{u - \sin u}{u^3}\,du\right\}$$

$$< \frac{4a^2k^2}{\sqrt{3}\,\pi^2}\left\{\int_{2ak}^{2}\frac{1}{2u}\,du + \int_{2}^{\infty}\frac{2}{u^3}\,du + \frac{\pi}{4}\right\} = \frac{4a^2k^2}{\sqrt{3}\,\pi^2}\left\{\frac{1}{2}\ln\frac{1}{ak} + \frac{1}{4} + \frac{\pi}{4}\right\}$$

$$< 0.25a^2k^2 + 0.12a^2k^2\ln\frac{1}{ak} \tag{4.33}$$

Combining inequalities (4.32) and (4.33) yields, on account of equation (4.31),

$$\left|\tilde{G}_1 - \tilde{C}_0 - ik\tilde{G}_{11}\right| < 0.63a^2k^2 + 0.12a^2k^2 \ln \frac{1}{ak}$$

from which the first of the estimates (4.30) follows.

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In an entirely analogous manner, since, for $\alpha > 0$

$$\left|e^{i\alpha} - 1\right| < \alpha$$

from the inequality

$$\tilde{G}_2 - \tilde{G}_{20} = \frac{4a}{3\pi^3}\int_{-a}^{a}\int_{\infty}^{\sqrt{a^2-y^2}}\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{\left(e^{ikx} - 1\right)\cos^2\gamma\sqrt{a^2 - y^2}\, d\gamma\, dx\, dy}{\sqrt{x^2 + y^2 - a^2}\left(x^2 + y^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma\right)}$$

the inequality is obtained

$$\left|\tilde{G}_2 - \tilde{G}_{20}\right| < \frac{2ka}{3\pi^2}\int_{-a}^{a}\int_{\sqrt{a^2-y^2}}^{\infty} \frac{\sqrt{a^2 - y^2}\, dy\, dx}{x\sqrt{x^2 + y^2 - a^2}} = \frac{2ka}{3\pi^2}\int_{-a}^{a}\int_{1}^{\infty}\frac{dy\, dt}{t\sqrt{t^2 - 1}} = \frac{2ka^2}{3\pi}$$

which proves the correctness of the second estimate (4.28).

The integral (4.30) was considered in reference 1. The function $H_1(\theta)$ of reference 1 is obtained if

$$\frac{3\pi^2}{2a}\sin\theta\, G_{20}\left\{-a\cos\theta\right) = H_1(\theta) \qquad (0 \leqslant \theta \leqslant \pi)$$

For this function the expression was obtained (equation (4.36) of reference 1 with the correction of the error appearing therein)

$$H_1(\theta) = \frac{3\pi}{2}\left\{\sin\theta\left(1 - \sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right) + \frac{\sin\theta}{12}\ln\frac{\left(1 + \cos\frac{1}{2}\theta\right)\left(1 + \sin\frac{1}{2}\theta\right)}{\left(1 - \cos\frac{1}{2}\theta\right)\left(1 - \sin\frac{1}{2}\theta\right)} + \right.$$

$$\left. \sin\theta\cos\theta\left[\ln\tan\frac{\theta}{2} + \frac{1}{4}\ln\frac{\left(1 + \cos\frac{1}{2}\theta\right)\left(1 - \sin\frac{1}{2}\theta\right)}{\left(1 - \cos\frac{1}{2}\theta\right)\left(1 + \sin\frac{1}{2}\theta\right)}\right]\right\} \qquad (4.34)$$

The expansion of this function in the interval $0 \leq \theta \leq \pi$ in a trigonometric series has the form

$$H_1(\theta) = \sum_{k=0}^{\infty} \gamma_{2k+1} \sin(2k + 1)\, \theta$$

where

$$\gamma_1 = \pi - \frac{17}{3} - \int_0^{\frac{1}{2}\pi} \ln \tan \frac{x}{2}\, dx = -0.69314$$

Hence

$$\tilde{G}_{20} = \frac{2}{\pi} \int_0^{\pi} G_{20}(-a \cos \theta) \sin^2 \theta\, d\theta = \frac{4a}{3\pi^3} \int_0^{\pi} \sin \theta\, H_1(\theta)\, d\theta = \frac{2a}{3\pi^2} \gamma_1 = -0.0468a \tag{4.35}$$

The mean value $\tilde{G}_{11}$ is computed. Integrating (4.29) with respect to γ yields

$$G_{11}(y) = \frac{a}{\pi^2} \int_{+\infty}^{\sqrt{a^2-y^2}} \frac{x}{\sqrt{x^2 + y^2 - a^2}} \left\{ -\frac{\pi x}{2(x^2 + y^2)} + \frac{y}{2(x^2 + y^2)} \ln \frac{x^2 + (y - a)^2}{x^2 + (y + a)^2} + \frac{x(a^2 + x^2 + y^2)}{(x^2 + y^2)(x^2 + y^2 - a^2)} \arctan \frac{x^2 + y^2 - a^2}{2ax} \right\} dx \tag{4.36}$$

If

$$x = at \qquad y = -a \cos \theta \qquad \frac{\pi^2}{a} \sin \theta\, G_{11}(-a \cos \theta) = H(\theta)$$

then

$$H(\theta) = \int_{0}^{\sin\theta} \frac{\sin\theta}{\sqrt{t^2 - \sin^2\theta}} \left\{ -\frac{\pi t^2}{2(t^2 + \cos^2\theta)} - \frac{t\cos\theta}{2(t^2 + \cos^2\theta)} \ln \frac{t^2 + 4\cos^4\frac{1}{2}\theta}{t^2 + 4\sin^4\frac{1}{2}\theta} + \right.$$

$$\left. \frac{t^2(t^2 + 1 + \cos^2\theta)}{(t^2 + \cos^2\theta)(t^2 - \sin^2\theta)} \arc\tan\frac{t^2 - \sin^2\theta}{2t} \right\} dt$$

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Computation of this integral results in

$$H(\theta) = \pi\left\{ \sin\theta\left(1 - \sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right) + \frac{1}{4}\sin\theta \ln\frac{\left(1 + \cos\frac{1}{2}\theta\right)\left(1 + \sin\frac{1}{2}\theta\right)}{\left(1 - \cos\frac{1}{2}\theta\right)\left(1 - \sin\frac{1}{2}\theta\right)} + \right.$$

$$\left. \frac{1}{4}\sin 2\theta\left[\ln\tan\frac{\theta}{2} - \ln\frac{1 + \sin\frac{1}{2}\theta}{1 + \cos\frac{1}{2}\theta}\right]\right\} \tag{4.37}$$

Further,

$$\tilde{G}_{11} = \frac{2}{\pi}\int_0^{\pi} G_{11}(-a\cos\theta)\sin^2\theta\, d\theta = \frac{2a}{\pi^3}\int_0^{\pi} H(\theta)\sin\theta\, d\theta$$

The computation of the last integral leads to the result

$$\tilde{G}_{11} = \frac{a}{\pi^2}\left\{\frac{2}{3}\pi - \frac{38}{9} + 2\int_0^{\frac{1}{2}\pi} \frac{u}{\sin u}\, du\right\} = 1.536\frac{a}{\pi^2} = 0.1556a \tag{4.38}$$

Thus for small ka

$$\tilde{G}_1 = 0.0947 + 0.1556ika \qquad \tilde{G}_2 = -0.0468a \tag{4.39}$$

Substituting these values in (4.26) gives

$$A = c\beta_1\frac{1 - 0.0936ika}{0.9053 - 0.2021ka} \qquad B = c\beta_1\frac{ik(0.8106 - 0.311ika)}{0.9053 - 0.2021ka} \tag{4.40}$$

Thus for small frequencies of vibration, to a first approximation:

$$A_0 = 1.105c\beta_0 \qquad A = (1.105 + 0.1441ka)\, c\beta_1$$

$$B = 0.8951kc\beta_1 \tag{4.41}$$

For the periodic vibrations with small frequency, in accordance with the law (4.1) of a plane circular wing, the previously derived formulas may be used for the forces where the values A_0, A, and B have the values just given. For the lift force, the approximate expression is obtained from equation (4.4)

$$P = \rho c^2a^2 \left\{2.813\beta_0 + \beta_1(2.813 \cos \omega t - 1.766ka \sin \omega t)\right\} \tag{4.42}$$

The fluctuation in the lift force due to the vibrations of the wing thus leads the latter in phase, the maximum value of the lift force being greater than the value which was obtained in the computation for the steady motion.

In the same way, equation (4.6) leads to the following expression for the moment of the pressure forces about the y-axis:

$$M_y = -\rho c^2a^3 \left\{1.473\beta_0 + \beta_1(1.473 \cos \omega t + 0.867\, ka \sin \omega t)\right\} \tag{4.43}$$

The component of the frontal resistance W_1 is determined in the given case by the evident formula

$$W_1 = P(\beta_0 + \beta_1 \cos \omega t)$$

that is,

$$W_1 = \rho c^2a^2 \left\{2.813\beta_0^2 + 1.406\beta_1^2 + \beta_0\beta_1(5.626 \cos \omega t - 1.766ka \sin \omega t) + 1.406\beta_1^2 \cos 2\omega t - 0.883\beta_1^2\, ka \sin 2\omega t\right\} \tag{4.44}$$

The suction force is obtained from equation (4.7), restricted to the first powers of ka,

$$W_2 = \rho c^2a^2 \left\{1.554\beta_0^2 + 0.777\beta_1^2 + \beta_0\beta_1(3.107 \cos \omega t + 1.888ka \sin \omega t) + 0.777\beta_1^2 \cos 2\omega t + 0.944ka\, \beta_1^2 \sin 2\omega t\right\} \tag{4.45}$$

The following expression is obtained for the total frontal resistance:

$$W = W_1 - W_2 = \rho c^2 a^2 \left\{1.259\beta_0^2 + 0.630\beta_1^2 + \beta_0\beta_1(2.519 \cos \omega t - 3.653ka \sin \omega t) + 0.630\beta_1^2 \cos 2\omega t - 1.827\beta_1^2\, ka \sin 2\omega t\right\} \tag{4.46}$$

For the mean value of the frontal resistance

$$\overline{W} = \rho c^2 a^2 \left\{1.259\beta_0^2 + 0.630\beta_1^2\right\} \tag{4.47}$$

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The flapping wing is considered such that

$$z = \beta_0 x + \beta_1 \cos \omega t \tag{4.48}$$

In this case

$$Z_0(x,y) = -c\beta_0 \qquad Z(x,y) = -ikc\beta_1 \tag{4.49}$$

Comparison of these expressions with equation (4.25) shows that in the case considered it is necessary to take

$$A_0 = \frac{c\beta_0}{1 - \tilde{G}_0} \qquad A = \frac{ikc\beta_1(1 + ik\tilde{G}_2)}{1 - \tilde{G}_1 + ik\tilde{G}_2} \qquad B = \frac{k^2 c\beta_1 \tilde{G}_1}{1 - \tilde{G}_1 + ik\tilde{G}_2} \tag{4.50}$$

that is,

$$A_0 = 1.105c\beta_0 \qquad A = ikc\beta_1 \frac{1 - ika\,0.0468}{0.9053 - 0.202ika} \qquad B = \frac{k^2 c\beta_1(0.0947 + 0.156ika)}{0.9053 - 0.202ika} \tag{4.51}$$

or, by restriction to small terms of the second order with respect to k,

$$A_0 = 1.105c\beta_0 \qquad A = ikc\beta_1(1.105 + 0.195ika) \qquad B = 0.105k^2 c\beta_1 \tag{4.52}$$

For the lift force

$$P = \rho c^2 a^2 \left\{2.813\beta_0 + 2.813k\beta_1 \sin \omega t + 0.301k^2 a\beta_1 \cos \omega t\right\} \tag{4.53}$$

and for the moment of the pressure forces about the y-axis

$$M_y = -\rho c^2 a^3 \left\{1.473\beta_0 + 1.473k\beta_1 \sin \omega t - 0.181k^2 a\beta_1 \cos \omega t\right\} \tag{4.54}$$

The component of the frontal resistance

$$W_1 = P\beta_0 = \rho c^2 a^2 \left\{ 2.813\beta_0{}^2 + 2.813k\beta_0\beta_1 \sin \omega t + 0.301k^2 a\beta_0\beta_1 \cos \omega t \right\} \tag{4.55}$$

The suction force will be, with an accuracy up to terms of the second order with respect to ka:

$$W_2 = \rho a^2 c^2 \left\{ 1.554\beta_0{}^2 + 0.777k^2\beta_1{}^2 - 0.376\beta_0\beta_1 k^2 a \cos \omega t + 3.107k\beta_0\beta_1 \sin \omega t - 0.777k^2\beta_1{}^2 \cos 2\omega t \right\} \tag{4.56}$$

For the total frontal resistance

$$W = \rho a^2 c^2 \left\{ 1.259\beta_0{}^2 a - 0.777k^2\beta_1{}^2 - 0.294k\beta_0\beta_1 \sin \omega t + 0.677k^2 a\beta_0\beta_1 \cos \omega t + 0.777k^2\beta_1{}^2 \cos 2\omega t \right\} \tag{4.57}$$

Its mean value will be

$$\overline{W} = \rho a^2 c^2 \left\{ 1.259\beta_0{}^2 - 0.777k^2\beta_1{}^2 \right\} \tag{4.58}$$

so that a decrease is obtained in the frontal resistance as compared with the wing which does not execute a flapping motion.

REFERENCES

1. Kochin, N. E.: Theory of a Wing of Circular Plan Form. Prikladnaya Matematika i Mekhanika, vol. IV, no. 1, 1940, pp. 3-32.

2. Schade, Th.: Theorie der schwingenden kreisförmigen Tragfläche auf potentialtheoretischer Grundlage. I Analytischer Teil. Luftfahrtforschung, bd. 17, lfg. 11/12, 1940, pp. 387-400.